Joint Distributions -Two or More RVs

Section 5.1

- So far we discussed only a single RV
- In real (useful) experiments, we usually collect information on two or more quantities simultaneously
  - Pressure, Volume, Temperature
  - Service Times, # of A/C units
  - # of cars and # of trucks on a Hwy
  - fuel consumption and average speed traveled
  - height and weight
  - time until failure of two different components
- We need to be familiar with probabilistic modeling of two or more RV's
  - Prediction of one variable given the values of the other(s)

Joint Distribution for Discrete RV's

Def: The joint pmf of X and Y is a function $p(x\ ,\ y)$ or $p_{XY}(x\ ,\ y)$ satisfying:

$$p(x,y) = P(X = x, Y = y)$$

Genuine joint pmf:

Let A be a set of (x,y) pairs. Find the P(A):

e.g. A = {(x,y): x + y = 5} or A = {(x,y): x ≥ 3, y ≤ 4}

Example 5.1

X = auto policy deductible

Y = homeowner's policy deductible

...for a randomly selected person in an insurance pool

Joint Probability Table

$p(x,y)$		$y = 0$	100	200
x	100	0.20	0.10	0.20
	250	0.05	0.15	0.30

Marginal Distributions

The marginal pmf's of X and Y, denoted by $p_X(x)$ and $p_Y(y)$ are

$$p_X(x) = \sum_y p(x,y) \quad \text{and} \quad p_Y(y) = \sum_x p(x,y)$$

$p(x,y)$		$y = 0$	100	200
x	100	0.20	0.10	0.20
	250	0.05	0.15	0.30

Example -Transportation

- The joint probability distribution (pmf) of the number of cars, X, and the number of buses, Y, per signal cycle at a proposed left turn lane (as a two way table)

	# buses		y		
# cars	p(x,y)	0	1	2	Col. Sum
	0	.025	.015	.01	.05
	1	.050	.030	.02	.10
x	2	.125	.075	.05	.25
	3	.150	.090	.06	.30
	4	.100	.060	.04	.20
	5	.050	.030	.02	.10
	Row Sum	.5	.3	.2	1.00

Transportation Example - Continued

a. Probabilities of some events of interest:
P(exactly one car *and* one bus during a signal cycle) = *p*(1,1) =

P(at most two cars *and* at most one bus) = $P(X \le 2, Y \le 1) = F(2,1) =$

P(fewer than two cars and at most one bus*)* = $P(X < 2, Y \le 1) = F(1,1) =$

P(one car) = $p_X(1) =$ *P*(no bus) = $p_Y(0) =$

Event of interest- Functions of two variables
Suppose that the left turn lane is designed for a capacity of five cars. Assume that one bus takes the space equivalent to that of three cars. Find the probability of an overflow during a signal cycle.

Transportation Example - Continued

b. Marginal distributions.

Marginal pmf of X:

Marginal pmf of Y:

c. Independence: Are X and Y independent random variables?

Note: Given only the marginal distributions, we can't find the joint distribution, unless we are told that X and Y are independent random variables.

d. WARNING: Small perturbations in joint probability, while keeping the same marginal distributions, can lead to the two variables being not independent.

Two Continuous RV's

Let X and Y be continuous RV's. A joint pdf f(x,y) for these two random variables is a function satisfying:

$$f(x,y) \geq 0 \quad \text{and} \quad \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dxdy = 1$$

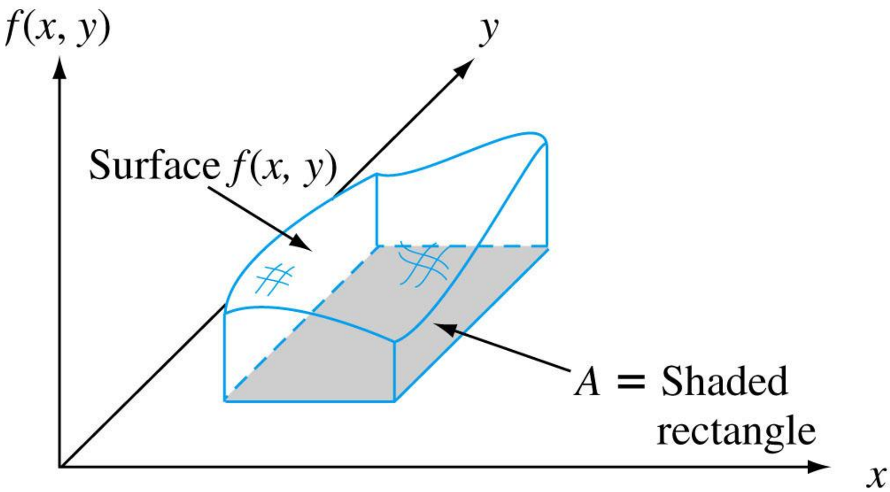
Let A be a region in two-dimensions:

Eg: A = rectangle = $\{(x,y): a \leq x \leq b,\ c \leq y \leq d\}$

CAUTION: This is NOT the area of the rectangle….

So, what is it?

Geometry



Marginal Distributions

$$f_X(x) = \int_{-\infty}^{\infty} f(x, y)\, dy$$

$$f_Y(y) = \int_{-\infty}^{\infty} f(x, y)\, dx$$

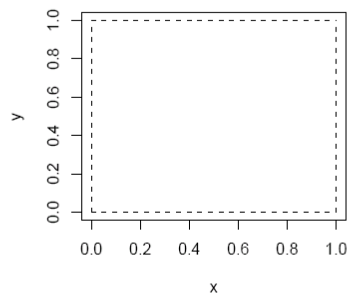
Example 5.3

Let X and Y have the joint density:

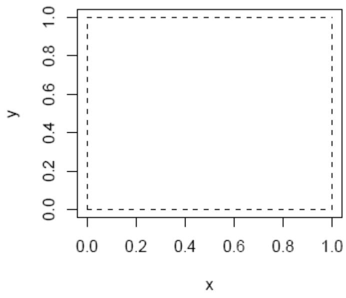
$$f_{XY}(x, y) = \begin{cases} k(x+y)^2 & 0 \le x \le 1, \quad 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

a) Find k

b) Find P (X>Y) by first identifying the region of integration and then performing the integral



c) Find $P(|X - Y| \leq 0.5)$ by first identifying the region of integration



d) Find the marginal distribution of X: $f_X(x)$

e) Marginal distribution of Y:

Jointly Distributed Random Variables

Section 5.1 cont.

Two Discrete RV's	Two Continuous RV's
pmf	**pdf**
$p(x,y) = P(X=x, Y=y)$	$f(x,y)$

computing probabilities

$$P((X,Y)\in A) = \sum_{(x,y)\in A} p(x,y) \qquad P((X,Y)\in A) = \iint_A f(x,y)\,dx\,dy$$

marginal distributions

$$p_X(x) = \sum_y p(x,y) \qquad f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy$$

$$p_Y(y) = \sum_x p(x,y) \qquad f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\,dx$$

Independence of Two Random Variables

Recall: Two events E_1 and E_2 are said to be *independent* if

$$P(E_1 \cap E_2) = P(E_1) \cdot P(E_2)$$

Otherwise, they are *dependent*.

Interpretation: Knowing whether E_1 has occurred does not tell us anything about whether E_2 has occurred, i.e., $P(E_1 \mid E_2) = P(E_1)$

Two discrete RV's are said to be *independent* if

$$p(x, y) = p_X(x) \cdot p_Y(y)$$

Example

X = auto policy deductible

Y = homeowner's policy deductible

Joint Probability Table

$p(x,y)$		$y=0$	$y=100$	$y=200$
x	100	0.20	0.10	0.20
	250	0.05	0.15	0.30

x	100	250
p(x)	0.5	0.5

y	0	100	250
p(y)	0.25	0.25	0.5

Two Continuous Random Variables

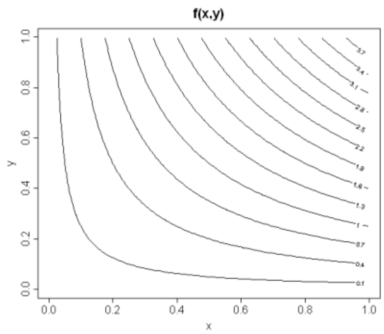
Two continuous RV's are said to be *independent* if

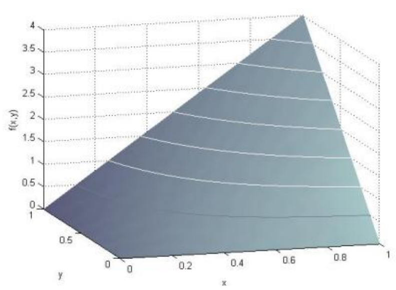
$$f(x,y) = f_X(x) \cdot f_Y(y)$$

Consequence of Independence:

Example

$$f(x,y) = 4xy \qquad \begin{matrix} 0 \leq & x & \leq 1 \\ 0 \leq & y & \leq 1 \end{matrix}$$





Example

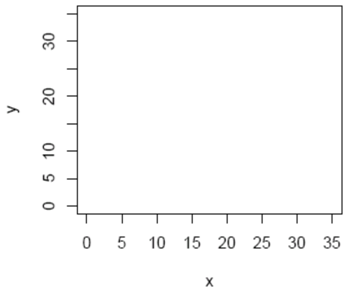
$$f(x,y) = 4xy \qquad \begin{array}{l} 0 \le x \le 1 \\ 0 \le y \le 1 \end{array}$$

Marginal Distributions:

Another Example

A health-food store stocks two different brands of a certain type of grain. Let *X* = the amount (in pounds) of brand A on hand and *Y* = the amount (in pounds) of brand B on hand. Suppose the joint pdf of *X* and *Y* is

$$f_{XY}(x,y) = \begin{cases} kxy & x \ge 0,\ y \ge 0,\ 20 \le x + y \le 30 \\ 0 & \text{otherwise} \end{cases}$$



a) First draw the region of positive density of and find the normalizing constant *k*.

b) Are *X* and *Y* independent?
Answer by first deriving the marginal pdf of each variable. **Hints:** The two marginal distributions are the same due to the symmetry of the problem (so you only need to compute one of the marginal distributions). Be careful: the marginal distributions are piecewise functions (there are two separate pieces you need to consider).

Note on Independence

Independent:

Not Independent:

Expected Values, Covariance, and Correlation

As seen earlier

Discrete

$$E[h(X)] = \sum_x h(x)\,p(x)$$

Continuous

$$E[h(X)] = \int h(x) f(x) dx$$

Let h(x,y) be a function of two variables (x,y).

When X and Y are **discrete** and jointly distributed with **pmf** p(x,y), then:

$$E[h(X,Y)] =$$

If X and Y are **continuous** and jointly distributed with **pdf** f(x,y), then:

$$E[h(X,Y)] =$$

Example

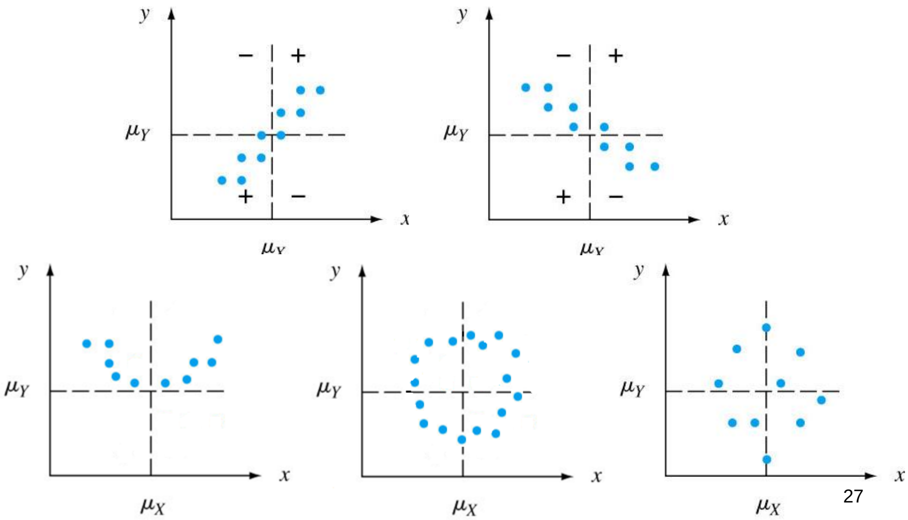
5.14 (modified) Suppose (example 5.5) the marginal distribution of amount of almonds (X) and the marginal distribution of the amount of cashews (Y) in a 1-lb can of nuts are:

$$f_X(x) = \begin{cases} 12x(1-x)^2 & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases} \qquad f_Y(y) = \begin{cases} 12y(1-y)^2 & 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

If 1 lb of almonds cost the company $1.00, 1 lb if cashews costs $1.50, and 1 lb of peanuts costs $0.50, then what is the expected total cost of a can?

Covariance

Covariance is a measure of **how strongly** two random variables are (linearly) related and whether this TREND is positive or negative



Covariance

$$\text{Cov}(X,Y) = E[(X-\mu_X)(Y-\mu_Y)] = E[XY] - \mu_X\mu_Y$$

Discrete Case:

Continuous Case:

Problem with interpreting **covariance** as a measure of the strength of relationship between X and Y:

$Cov\,(aX+b, cY+d) =$

Correlation

$$\rho = \rho_{X,Y} = \frac{\text{Cov}(X,Y)}{\sigma_X \cdot \sigma_Y}$$

Correlation is a measure of how strongly two random variables are **linearly** related

- $-1 \leq \text{Corr}(X, Y) \leq 1$
- If ac > 0, Corr (aX + b, cY + d) = Corr (X, Y)
- If X and Y are independent, then ρ = 0, BUT
  ρ = 0 does not imply independence.
- ρ = 1 or ρ = -1 if and only if Y = a X + b for some a ≠ 0

29

Transportation Example - Continued

e. Expected Values and Variances of X, Y
\# of Cars in one signal cycle, E(X), Var(X)

\# of Buses in one signal cycle, E(Y), Var(Y)

f. Cov(X.Y) and Cor(X,Y)
Find E(XY):

Cov(X,Y)=

Cor(X,Y)=