Buildings for generalized braid groups

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by K. Goldman (Ohio State University), 2023

Introduction

The main result of this work is the following.

Theorem. *Let V be a finite dimensional real vector space, $\mathscr{M}$ a finite set of homogeneous hyperplanes of V, $V_{\mathbb{C}}$ the complexification of V, and $Y = V_{\mathbb{C}} - \bigcup_{M \in \mathscr{M}} M_{\mathbb{C}}$. Suppose that the connected components of $V - \bigcup_{M \in \mathscr{M}} M$ are open simplicial cones. Then Y is a $K(\pi, 1)$.*

Take V as above and $W \subset GL(V)$ a finite group generated by reflections. Suppose no non-zero vector of V is fixed by W:

$$V^W = 0.$$

Let Φ be a Euclidean structure on V invariant under W, with $\mathscr{M}$ a set of hyperplanes M such that the orthogonal reflection with respect to M is in W. We then know that $(V, \mathscr{M})$ satisfies the hypothesis of the theorem, and that W acts freely on the corresponding space Y_W. The quotient $X_W = Y_W/W$ is therefore also a $K(\pi, 1)$.

This result had been conjectured by Brieskorn. It is only new for W of type H_3, H_4, E_6, E_7 and E_8 (Brieskorn [2]). We also give a new proof that the fundamental group of X_W is the generalized braid group $\tilde{W}$ corresponding to W (Brieskorn [3]).

Reduced to the particular case considered above, the outline of the proof is as follows.

a) We construct a building $I(\tilde{W})$ on which W acts, and a set $\mathscr{S}$ of spheres in $I(\tilde{W})$ isomorphic to the unit sphere of V. The group $\tilde{W}$ acts strictly transitively on $\mathscr{S}$.

b) We show that, up to homotopy, $I(\tilde{W})$ is the bouquet of the spheres $S \in \mathscr{S}$.

c) We relate X_W and $I(\tilde{W})$.

The description of $\mathscr{S}$ and the possibility of c) came from a conversation with Brieskorn and Tits in the spring of 1970. The ideas required to establish b) were provided to me by Garside [4], to whom I am often very near.

In Section 4, we determine the center of $\tilde{W}$, and we solve the word problem and conjugacy problem in $\tilde{W}$. These results were independently obtained by E. Brieskorn and K. Saito, who use Garside's methods as we do [4].

The geometric language used and the proof techniques are essentially due to Tits, whose ideas I was able to familiarize myself with by attending one of his seminars and during conversations.

I am happy to be able to express my gratitude to him here.

0. Notation

(0.1) $L^+(D)$: the free (unitial) monoid generated by the set D.

(0.2) $L(D)$: the free group generated by the set D.

(0.3) A^-: the closure of a subset A of a topological space.

(0.4) For elements x, y of a monoid and $n \geqq 0$, we write $\operatorname{prod}(n; x, y) = xyxy\ldots$ (n factors); we have $\operatorname{prod}(2n; x, y) = (xy)^n$, $\operatorname{prod}(2n+1; x, y) = (xy)^n x$.

§ 1. Galleries

(1.1) Let $\mathscr{M}$ be a finite set of hyperplanes (the *walls*) of a finite dimensional real vector space V. We use the terminology (*walls, chambers, faces, facets*) of [1] V §1 (the facets form a partition of E). The *support* P of a facet F is the intersection of the walls containing F; F is open in P. For any chamber A and any wall M, we denote by $D_M(A)$ the set of chambers on the same side of M as A. If A and B are two chambers, we denote by $D(A, B)$ the intersection of all $D_M(C)$ containing A and B [*tr. note: this definition has been corrected from the original paper*]. Leaving the terminology of [1] IV 1 ex 15, we say that two chambers A and B are *adjacent* if $A \neq B$ but A and B have one face in common. We will repeatedly use the following trivial facts.

(1.2) **Lemma.** (i) *Let M be a wall of a chamber B. There exists exactly one chamber B' adjacent to B which has M as a wall. M is the only wall which separates B and B'.*

(ii) *Let B_1, B_2, B_3 be three chambers, and $\mathscr{M}(B_i, B_j)$ the set of walls which separate B_i from B_j. We have*

$$\mathscr{M}(B_1, B_3) = \big(\mathscr{M}(B_1, B_2) - \mathscr{M}(B_2, B_3)\big) \cup \big(\mathscr{M}(B_2, B_3) - \mathscr{M}(B_1, B_2)\big)$$

A *gallery of length n* ($n \geqq 0$) *with start A and end B* is a sequence of chambers $(C_0, \ldots, C_n)$ with $A = C_0$, C_{i+1} adjacent to C_i ($0 \leqq i \leqq n$) and $C_n = B$. The *composition* of two galleries $G = (C_0, \ldots, C_n)$ and $G' = (C'_0, \ldots, C'_n)$ is defined if $C_n = C'_0$, in which case it is

$$GG' = (C_0, \ldots, C_n, C'_1, \ldots, C'_m).$$

If G is a gallery starting at A, $A.G$ denotes its end.

The opposite gallery G^* of a gallery $G = (C_0, \ldots, C_n)$ is the gallery $(C_n, \ldots, C_0)$. We have $(GG')^* = G'^*G^*$.

The *antipodal gallery* $-G$ of a gallery $G = (C_0, \ldots, C_n)$ is the sequence of antipodal chambers $-G = (-C_0, \ldots, -C_n)$.

We have $-(GG') = (-G)(-G')$ and $-(G^*) = (-G)^*$.

The *distance* $d(A, B)$ between a chamber A and chamber B is the smallest length of a gallery starting at A and ending at B. A gallery from A to B is *minimal* if its length is $d(A, B)$.

(1.3) **Proposition.** *The distance $d(A, B)$ is the number of walls which separate A and B. For a gallery G from A to B to be minimal, it is necessary and sufficient that it crosses the walls which separate A and B exactly once and does not cross any other walls.*

It is clear that any gallery from A to B crosses each wall which separates A and B, and it suffices to prove the existence of a gallery G from A to B of length equal to the number k of walls which separate A from B. We proceed by induction on k. The intersection of the $D_M(A)$ for M a wall of A is just A. Hence, either $A = B$, in which case we take $G = (A)$, or there exists a wall M of A which separates A from B. Let A' be the chamber adjacent to A with M as a wall. M is the only wall that separates A from A', so a wall N separates A' and B if and only if $M \ddagger N$ and separates A and B. According to the induction hypothesis, there exists a gallery G' of length $k - 1$ from A' to B, and we take $G = (AA')G'$.

(1.4) **Corollary.** *Let A, B, C be three chambers. The following conditions are equivalent.*

(i) $d(A, C) = d(A, B) + d(B, C)$, *i.e. there exists a minimal gallery from A to C passing through B.*

(ii) *For a wall to separate A from C, it is necessary and sufficient that it separate A from B or B from C. In other words, $\mathcal{M}(A, B) \subset \mathcal{M}(A, C)$* (cf. 1.2 (ii)).

(iii) $B \in D(A, C)$.

(1.5) **Proposition.** *Let P be an intersection of walls, $V_P = V/P$,* pr_P : $V \longrightarrow V_P$ *the projection map and $\mathcal{M}_P$ the collection of hyperplanes N of V_P such that* $\mathrm{pr}_P^{-1}(N)$ *is a wall. We denote by π'_P the unique map from the facets of $(V, \mathcal{M})$ to those of $(V_P, \mathcal{M}_P)$ satisfying* $\mathrm{pr}_P(F) \subset \pi'_P(F)$.

(i) π'_P *respects the incidence relation $F_1 \subset F_2^-$: if $F_1 \subset F_2^-$, we have*

$$\mathrm{codim}(F_1 \textit{ in } F_2^-) \geqq \mathrm{codim}(\pi'_P[F_1] \textit{ in } \pi'_P[F_2]).$$

π'_P takes chambers to chambers; if A and B are adjacent, then either $\pi'_P(A) = \pi'_P(B)$, or $\pi'_P(A)$ and $\pi'_P(B)$ are adjacent.

(ii) *Suppose F is a facet with support P* (1.1). *The restriction of π'_P to the set of facets E of $(V, \mathscr{M})$ such that $F \subset E^-$ is bijective. Thus the inverse π_F respects codimension, incidence $E_1 \subset E_2^-$, and therefore adjacency of chambers. We will also use π_F to denote the bijection* pr_P^{-1} *between intersections of walls in V_P and intersections of walls in V containing P.*

(iii) *If $C = \pi_F(C')$, we have, for each chamber X of $(V, \mathscr{M})$,*

$$\pi_F^{-1} D(C, X) = D\big(C', \pi'_P(X)\big) \quad \text{and} \quad \pi_F^{-1} \mathscr{M}(C, X) = \mathscr{M}_P\big(C', \pi'_P(X)\big).$$

For $X = \pi_F(X')$, we have

$$\pi_F\big(D(C', X')\big) = D(C, X) \quad \text{and} \quad \pi_F\big(\mathscr{M}_P(C', X')\big) = \mathscr{M}(C, X);$$

π_F then induces a bijection between minimal galleries from C' to X' and C to X.

The proof is left to the reader $\big($(iii) follows from 1.4(ii)$\Leftrightarrow$(iii)$\big)$.

(1.6) **Notation.** (i) For a facet F with support P, we let V_P, $\mathscr{M}_P$, pr_P, π'_P denote V_F, $\mathscr{M}_F$, pr_F, π'_F.

(ii) Suppose P is an intersection of walls and C is a chamber. We assume that there is a facet of C (i.e. in C^-) with support P. This facet is then unique. We denote it by $F(P)$ and set

$$C.\Delta(P) = \pi_{F(P)}\big(-\pi_{F(P)}^{-1}(C)\big).$$

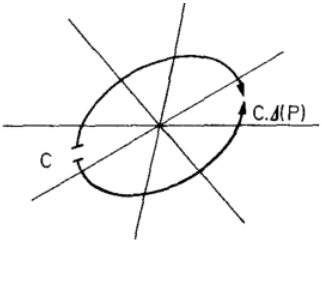
(1.7) **Lemma.** (i) *In order for a wall to separate C and $C.\Delta(P)$, it is necessary and sufficient that it contain P* $\big($(1.5)(iii)$\big)$.

(ii) *For a wall M of C, $C.\Delta(M)$ is the unique chamber adjacent to C with M as a wall.* $\big($cf. 1.5(iii), 1.2(i)$\big)$

(iii) *Suppose $M \neq N$ are two walls of C, whose intersection P contains a facet of C open in P. There are exactly two minimal galleries from C to $C.\Delta(P)$. One begins with $(C, C.\Delta(M))$, the other with $(C, C.\Delta(N))$.*

The assertion (iii) is shown by reduction to rank two, (cf. 1.5(iii)), where the drawing is

(1.7.1)



We will now assume that the following condition holds.

(1.8) **Assumption.** *The chambers are open simplicial cones.*

In other words, each chamber is a set of points with positive coordinates in a suitable basis of V.

(1.9.1) *Remark.* Let P be an intersection of walls.

(i) $(V_P, \mathscr{M}_P)$ also satisfies (1.8);

(ii) the set $\mathscr{M}^P$ of intersections with P of walls which don't contain P also satisfies (1.8).

If $\mathscr{M}$ is defined by a "Weyl group" W (see the introduction), $\mathscr{M}^P$ is not in general of this type.

(1.9.2) *Remark.* The hypothesis (1.8) guarantees that if P is an intersection of walls of a chamber C, the condition in (1.6) is satisfied. The chamber $C.\varDelta(P)$ is therefore defined.

(1.10) **Definition.** *Two galleries G and G' with the same beginning and end are equivalent (notation: $G \sim G'$) if there is a sequence of galleries $G = G_0, \ldots, G_n = G'$ ($n \geqq 0$) such that G_{j+1} is related to G_j in the following way ($0 \leqq j < n$):*

a) *we have decompositions $G_j = E_1 F E_2$, $G_{j+1} = E_1 F' E_2$;*

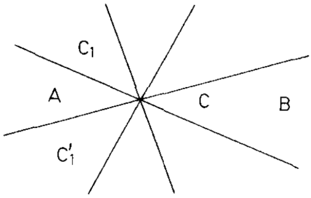
b) *there is a chamber C and two walls M, N of C such that F and F' are the two minimal galleries from C to $C.\varDelta(M \cap N)$.*

Composition of galleries, the operations $G \to G^*$ and $G \to -G$, the start and end, and the length function are all compatible with this equivalence relation and thus pass to the quotient. In general, we will denote gallery classes in lowercase. We say that a class of galleries e *begins* (resp. *ends*) with a class of galleries f if there exists g such that $e = fg$ (resp. $e = gf$). One checks:

(1.11) **Proposition.** *Two equivalent galleries cross each wall the same number of times.*

(1.12) **Proposition.** *Two minimal galleries with the same beginning and end are equivalent.*

Let $G = (C_0, \ldots, C_n)$ and $G' = (C'_0, \ldots, C'_n)$ with ends A and B. We proceed by induction over the length of the galleries. The case $n = 0$ is trivial and the induction hypothesis allows us to suppose that $C_1 \neq C'_1$. Let M and M' be the walls which separate $A = C_0 = C'_0$ from C_1 and C'_1, and $C = A.\varDelta(M \cap M')$. The walls M and M' separate A from B. By reduction to rank two (1.5)(iii), we deduce that any wall that separates A from C separates A from B.



$\big($the drawing in $V_{M\cap M'}$; we omit writing $\pi'_{M\cap M'}(\;)\big)$

Let F (resp. F') be the minimal gallery from A to C which passes through C_1 (resp. C'_1), and E a minimal gallery from C to B. The galleries FE and $F'E$ are minimal and equivalent. The minimal galleries G and FE (resp. G' and FE') start with (A, C_1) (resp. with (A, C'_1)). The induction hypothesis implies that they are equivalent, and (1.12) follows.

(1.13) **Notation.** (i) We denote by $u(A, B)$ the equivalence class of minimal galleries from A to B.

(ii) For I a set of walls in a chamber C, with intersection P, we set $\Delta(P) = u\big(C, C.\Delta(P)\big)$ (cf. (1.9.2)). For I the set of all walls, we set $\Delta = u(C, -C)$.

The benefit of this notation is in the ambiguity of C.

(1.14) **Proposition.** *Let A be a chamber and $\mathscr{G}$ a set of classes of galleries with bounded length starting at A. Suppose $\mathscr{G}$ satisfies* (i)*, the combination of*

($\mathrm{i_a}$) $(A) \in \mathscr{G}$.

($\mathrm{i_b}$) *If $gh \in \mathscr{G}$, then $g \in \mathscr{G}$.*

($\mathrm{i_c}$) *Suppose g ends at B and M, N are walls of B. If $g\Delta(M)$ and $g\Delta(N)$ are in $\mathscr{G}$, then $g\Delta(M\cap N) \in \mathscr{G}$.*

Then we have

(ii) *There is a (unique) class of galleries x starting at A such that*

$$\mathscr{G} = \{\, g \mid x \text{ starts with } g \,\}\,.$$

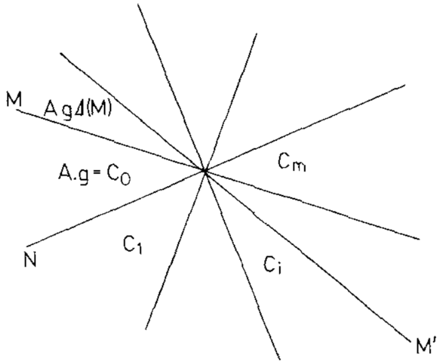
Uniqueness is clear: if x begins with y such that $x \neq y$, y is of length strictly less than x. Let x be of maximum length in $\mathscr{G}$. To be sure that this is the correct choice of x, it suffices to prove the following assertion.

($*$) Suppose g and M are such that (a) x starts with g and (b) x does not start with $g\Delta(M)$, and $g\Delta(M) \in \mathscr{G}$. There also exist g' and M' satisfying (a) and (b), with g' strictly longer than g.

Since $g\Delta(M) \in \mathscr{G}$, the maximality of x implies that $g \neq x$, therefore x begins with $g\Delta(N)$ for suitable N, necessarily distinct from M. From ($\mathrm{i_c}$), we then have $g\Delta(M\cap N) \in \mathscr{G}$.

Since $g\Delta(M\cap N)$ starts with $g\Delta(M)$, x does not start with $g\Delta(M\cap N)$.

Let $(C_0,\ldots,C_m)$ be the minimal gallery from $A.g$ to $A.g\Delta(M\cap N)$ starting with $(A.g, A.g\Delta(N))$. Let i $(0<i<m)$ be the largest i such that x starts with $g'=g(C_0,\ldots,C_i)$. Then, g' and the wall M' between C_i and C_{i+1} satisfy (a)(b).



(projection onto $V_{M\cap N}$).

(1.15) **Proposition.** *Let A be a chamber and $\mathcal{B}$ a set of chambers. In order for $\mathcal{B}$ to satisfy* (i)*, the combination of*

(i$_a$) *$A\in\mathcal{B}$.*

(i$_b$) *If $B\in\mathcal{B}$, then $D(A,B)\in\mathcal{B}$.*

(i$_c$) *Suppose M and N are walls of a chamber B. If A and B are on the same side of M and N, and if $B.\Delta(M)$ and $B.\Delta(N)$ are in $\mathcal{B}$, then $B.\Delta(M\cap N)\in\mathcal{B}$.*

it is necessary and sufficient that

(ii) *There is a (unique) chamber C such that $\mathcal{B}=D(A,C)$.*

It is easily verified that (ii) ⇒ (i). To prove that (i) ⇒ (ii), one uses (1.14) to the set $\mathcal{G}$ of $u(A,B)$ for $B\in\mathcal{B}$ (note that by (1.3) and (1.11) any gallery G in the class g such that $u(A,B)$ begins with g is automatically minimal) and we apply (1.4)(i)⇔(iii) to get the class of galleries $x=u(A,X)$.

Applying criterion (1.15), we find:

(1.16) **Corollary.** *Let A and B be adjacent chambers separated by a wall M, and C a chamber on the same side of M as B. There is a C' such that*

$$D(A,C)\cap D_M(B)=D(B,C').$$

(1.17) **Corollary.** (i) *Let A, C_1, C_2 be three chambers. There is a C such that*

(1.17.1) $$D(A,C)=D(A,C_1)\cap D(A,C_2).$$

(ii) *Let A and C_1 be two chambers with M a wall of A. There is a C' such that*

(1.17.2) $$D(A, C') = D(A, C_1) \cap D_M(A).$$

In fact, (ii) is the special case of (i) for $C_2 = -(A.\varDelta(M))$.

(1.18) **Lemma.** *Let M, M', M'' be three distinct walls of a chamber A, $A_1 = A.\varDelta(M)$, $B = A.\varDelta(M \cap M' \cap M'')$, $B' = A.\varDelta(M \cap M')$, and $B'' = A.\varDelta(M \cap M'')$. For any chamber C, if $B' \in D(A_1, C)$ and $B'' \in D(A_1, C)$, we have $B \in D(A_1, C)$.*

The facet F of A open in $M \cap M' \cap M''$ is a facet of all the chambers A, A_1, B, B', and B''. Applying (1.5)(iii), we can therefore go back to the case where $\dim V = 3$. Applying (1.17)(ii), we can assume furthermore that A_1 and C are on the same side of M. Let us make these assumptions, and suppose that $B', B'' \in D(A_1, C)$. Since M' (resp. M'') separates A_1 from B' (resp. B''), A_1 and C are separated by M', M'' (and on the same side of M); A and C are therefore separated by M, M' and M'', and $C = -A = B$.

The following key result is inspired by [4].

(1.19) **Proposition.** (i) *Let A, B, C be chambers, F a gallery from A to B, and G_1, G_2 two galleries from B to C. If $FG_1 \sim FG_2$, then $G_1 \sim G_2$.*

(ii) *Likewise, if $G_1E \sim G_2E$, then $G_1 \sim G_2$.*

(iii) *Suppose g is an equivalence class of galleries starting at A. There is a chamber C such that g starts with $u(A, B)$ if and only if $B \in D(A, C)$.*

One can prove (ii) from (i) by passing to opposite galleries.

Let $G = (C_0, \dots, C_n)$ and $G' = (C'_0, \dots, C'_n)$ be two galleries of length n with the same ends. If $n \geqq 1$, we set $G_1 = (C_1, \dots, C_n)$ and $G'_1 = (C'_1, \dots, C'_n)$. If moreover $C_1 \neq C'_1$, we let M and M' be the walls which separate $C_0 = C'_0$ from C_1 and C'_1, and we set $A = C_0.\varDelta(M \cap M')$. Consider the following relation $R_n(G, G')$ between galleries G, G' as above.

$R_n(G, G') \Leftrightarrow$ we have either

α) $n = 0$;

β) $n \neq 0$, $C_1 = C'_1$, and $G_1 \sim G'_1$;

γ) $n \neq 0$, $C_1 \neq C'_1$, and there is a gallery F from A to $C_n = C'_n$ such that $G_1 \sim u(C_1, A).F$ and $G'_1 \sim u(C'_1, A).F$.

We will prove the following assertion by induction on n.

(A_n) R_n is an equivalence relation.

Let us assume (A_i) for $i < n$ and prove (1.19.1) through (1.19.3) below.

(1.19.1) For galleries of length $i < n$, $R_i(G, G') \Leftrightarrow G \sim G'$.

It's trivial that $R_i(G, G') \Rightarrow G \sim G'$, and, with the notation of (1.10), if $G \sim G'$, we have $R_i(G_j, G_{j+1})$.

(1.19.2) The assertion (i) for FG_1 of length $< n$ holds.

This follows from (1.19.1) and the second case of the definition of R_i.

(1.19.3) The assertion (iii) holds for g of length $< n$.

This follows from (1.15) applied to the set $\mathscr{B}$ of chambers B such that g starts with $u(A, B)$: The condition ($\mathrm{i_c}$) of (1.15) is shown to hold with the help of (1.19.2), of (1.18), and the third clause in the definition of R.

It remains to prove that if G, G' and G'' are three galleries of length n from A to C and that $R_n(G, G')$ and $R_n(G, G'')$, we have $R_n(G', G")$. If $n = 0$, $C_1 = C'_1$, or $C_1 = C''_1$, this is trivial.

For $n > 0$, let M, M', M'' denote the walls which separate $A = C_0 = C'_0 = C''_0$ from C_1, C'_1, and C''_1. We distinguish two cases

Case 1. $C_1 \neq C'_1 = C''_1$. Let $B = A.\Delta(M \cap M')$. By our assumptions,

$$\begin{aligned}
&(C_1, \ldots, C_n) \sim u(C_1, B)F' \sim u(C_1, B)F'',\\
&(C'_1, \ldots, C'_n) \sim u(C_1, B)F',\\
&(C''_1, \ldots, C''_n) \sim u(C''_1, B)F''.
\end{aligned}$$

From (1.19.2), we conclude that $F' \sim F''$, and therefore that $R_n(G', G'')$.

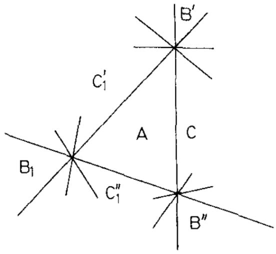
Case 2. $C_1 \neq C'_1 \neq C''_1 \neq C_1$. Let $B_1 = A.\Delta(M' \cap M'')$, $B' = A.\Delta(M \cap M')$, $B'' = A.\Delta(M \cap M'')$ and $B = A.\Delta(M \cap M' \cap M'')$. We have

$$\begin{aligned}
G_1 &= (C_1, \ldots, C_n) \sim u(C_1, B)F' \sim u(C_1, B)F'',\\
G'_1 &= (C'_1, \ldots, C'_n) \sim u(C_1, B)F',\\
G''_1 &= (C''_1, \ldots, C''_n) \sim u(C''_1, B)F''.
\end{aligned}$$

so that the class g_1 of G_1 starts with $u(C_1, B')$ and $u(C_1.B'')$. It then follows from (1.19.3) and Lemma (1.18) that g_1 starts with $u(C_1, B)$: $G_1 \sim u(C_1, B)F$. From (1.19.2), we have $F' \sim u(B', B)F$ and $F'' \sim u(B'', B)F$, thus $G_1 \sim u(C_1, B)F$, $G'_1 \sim u(C'_1, B)F$ and $G''_1 \sim u(C''_1, B)F$. Therefore,

$$\begin{aligned}
G'_1 &\sim u(C'_1, B_1)\, u(B_1, B)\, F\\
G''_1 &\sim u(C''_1, B_1)\, u(B_1, B)\, F,
\end{aligned}$$

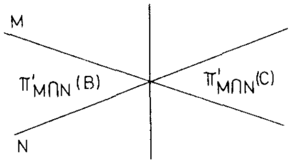
so $R_n(G', G'')$.



(picture in the unit sphere of $V_{M \cap M' \cap M''}$).

(1.20) **Corollary.** *Conditions* (i) *and* (ii) *of* (1.14) *are equivalent.*

We suppose (ii) and prove (i_c). Under the hypotheses of (i_c), if $x = gh$ (h starting at B), it follows from (1.19)(i) that h begins with $\Delta(M)$ and $\Delta(N)$. Let C be the chamber guaranteed by (1.19)(iii) for h. Since M and N separate B from C, $\pi'_{M\cap N}(C)$ can only be $\pi'_{M\cap N}(B).\Delta$ and (i_c) follows from (1.5)(iii).



(1.21) **Corollary.** *For any chamber A, let $n(A,i)$ be the number of classes of galleries starting at A with length i. Set*

$$f_A = \sum_0^\infty n(A,i)t^i \in \mathbb{Z}[[t]].$$

For each chamber A, and for P ranging over the intersection of walls of A (including $\{0\}$ and V), we have

$$\sum_P (-1)^{\operatorname{codim}(P)} t^{d(A,A.\Delta(P))} f_{A.\Delta(P)} = 1.$$

This means that

a) the number of classes of galleries of length i starting at $\Delta(P)$ is $n(A.\Delta(P), i - d(A, A.\Delta(P)))$. $\big((1.19)(iii)\big)$.

b) the classes of galleries which start at $\Delta(P)$ and $\Delta(Q)$ begin with $\Delta(P\cap Q)$ $\big($follows from (1.19)(iii)$\big)$.

(1.22) *Algorithm.* Let $G = (A_0, \dots, A_n)$ be a gallery of length $n \geqq 1$, $G_1 = (A_1, \dots, A_n)$, M the wall which separates A_0 and A_1, C the chamber whose existence is guaranteed by (1.19)(iii) (for G) and C_1 the analogous chamber for G_1. We can easily compute C by induction with the following formula $\big($cf. (1.16)$\big)$

(1.22.1) $$D(A_1, C) = D_M(A_1) \cap D(A_1, C_1).$$

Since $A_1 \in D(A,C)$, M indeed separates A from C and $C \in D_M(A_1)$. According to (1.19)(i), we also have $D(A_1,C) \subset D(A_1,C_1)$, whence the inclusion $\subset$. Finally, if $B \in D_M(A_1) \cap D(A_1,C_1)$, G_1 starts with $u(A_1,B)$ and G starts with $u(A,A_1).u(A_1,B) = u(A,B)$, whence (1.22.1).

(1.23) **Corollary.** *Let G and H be two composable galleries. Let A be the start of G, B that of H and G guaranteed by (1.19)(iii): we have $H \sim u(B,C).H'$. So, for any chamber D, for the class of GH to start with $u(A,D)$, it is necessary (and sufficient) that $Gu(B,C)$ start with $u(A,D)$.*

(1.24) **Proposition.** *Let A be a chamber, P an intersection of walls of A, $F = F(P)$ (1.6) and g an equivalence class of galleries starting at A. There exists a class of galleries g' of $(V_P, \mathscr{M}_P)$, of source $\pi_F^{-1}(A)$, such that for each gallery H of $(V_P, \mathscr{M}_P)$, g starts with $\pi_F(H)$ is and only if g' starts with H.*

Let $\mathscr{G}$ be a set of classes of galleries h of source $\pi_F^{-1}(A)$ in $(V_P, \mathscr{M}_P)$ such that g starts with $\pi_F(h)$. We apply (1.14) to $(V_P, \mathscr{M}_P)$ and to $\mathscr{G}$. Condition (i) from (1.19) is satisfied (cf. the proof of (1.20)), and (1.24) follows from (1.14)(ii).

(1.25) We can regard the set of galleries as the set of arrows of a category $\mathrm{Gal}_0(V, \mathscr{M})$ with the chambers for objects. Likewise, the equivalence classes of galleries are the arrows of a quotient category $\mathrm{Gal}_+(V, \mathscr{M})$. Let A, B, and C be three chambers, E a gallery from A to B, and F a gallery from B to C. Although the general conventions in categories are different, we continue to let EF denote the composition of E and F. The law $*$ (resp. $G \to -G$) is an antiequivalence (resp. an equivalence) of $\mathrm{Gal}_0(V, \mathscr{M})$ or $\mathrm{Gal}_+(V, \mathscr{M})$ with itself, inducing the identity (resp. $C \to -C$) on the set of objects.

When no confusion can be had, we write simply Gal_0 and Gal_+ for $\mathrm{Gal}_0(V, \mathscr{M})$ and $\mathrm{Gal}_+(V, \mathscr{M})$ (or for a category $\mathrm{Gal}_0(V_P, \mathscr{M}_P)$ or $\mathrm{Gal}_+(V_P, \mathscr{M}_P)$). By abuse of language, we often call *galleries* the arrows of Gal_+.

(1.26) **Lemma.** (i) *In* Gal_+, *for each gallery g, we have*

$$g\Delta = \Delta(-g).$$

(ii) *For each g and h starting at A in* Gal_+, *there exists an n such that $g\Delta^n$ starts with h. If h is composed of k galleries $u(A_i, A_{i+1})$, we can take $n = k$.*

Proceeding by induction, we first prove (i) for g of length one. For $g = (B, C)$, we have

$$\begin{aligned} g\Delta &= u(B,C)u(C,-C) = u(B,C)u(C,-B)u(-B,-C) \\ &= u(B,-B)u(-B,-C) = \Delta(-g). \end{aligned}$$

For each chamber B, $\Delta = u(A, -A)$ starts with $u(A, B)$. Hence, (ii) can be deduced from (i) by induction on k.

The following proposition follows immediately from (1.26), via transformation by $*$ and (1.19)(i), (ii).

(1.27) **Proposition.** (i) *In* Gal_+, *the set of all arrows gives rise to a calculation of left and right fractions.*

(ii) *Let* Gal$(V, \mathscr{M})$, *or simply* Gal, *be the category (a groupoid) derived from* Gal$_+$ *by adding inverses to each arrow: we call images of arrows of* Gal$_+$ *in* Gal *positive. The canonical functor from* Gal$_+$ *to* Gal *is faithful, and each arrow g of* Gal *can be put in the form $g = g_1 \varDelta^{-n} = \varDelta^{-n} g_2$ with g_1 and g_2 positive.*

If a category C has only one object A and if the monoid $\mathrm{Hom}(A, A)$ is reduced, then the set of all arrows of C admits a calculation of left and right fractions, meaning that $\mathrm{Hom}(A, A)$ satisfies the left and right condition of Öre. The "theory of calculating fractions" is used in (1.27) and below is an immediate generalization of the theory of Öre to embed a monoid in a group.

(1.28) For each wall M, the number of times each g in Gal$_+$ crosses M is well defined (1.11). This function of g is extended by additivity to $g \in$ Gal. More generally, for each intersection P of walls, we may use the universal property of Gal to define functors

$$\pi'_P : \mathrm{Gal}(V, \mathscr{M}) \to \mathrm{Gal}(V_P, \mathscr{M}_P).$$

(1.29) Let A be a chamber, P an intersection of walls of A, and $F = F(P)$ (1.6). The function π_F induces a functor

$$\pi_F : \mathrm{Gal}_0(V_P, \mathscr{M}_P) \to \mathrm{Gal}_0(V, \mathscr{M}). \tag{1.29.1}$$

The image of this functor is stable under equivalence, and it induces a *faithful* functor

$$\pi_F : \mathrm{Gal}_+(V_P, \mathscr{M}_P) \to \mathrm{Gal}_+(V, \mathscr{M}). \tag{1.29.2}$$

From the theory of the calculus of fractions and from (1.19)(i), we have

(1.30) **Proposition.** *Under the previous hypotheses, the functor deduced from* (1.29.2)

$$\pi_F : \mathrm{Gal}(V_P, \mathscr{M}_P) \to \mathrm{Gal}(V, \mathscr{M})$$

is faithful.

(1.31) **Proposition.** *Let C be a chamber, I and J two sets of walls of C, $K = I \cap J$, P, Q, and R the intersections of the walls of I, J, and K, and $F(P)$, $F(Q)$, and $F(R)$ the corresponding facets in C (1.6). The facet $F(R)$ is the smallest facet containing $F(P)$ and $F(Q)$. In the groupoid* Gal$(V, \mathscr{M})$, *we have*

$$\pi_{F(P)}\big(\mathrm{Gal}(V_P, \mathscr{M}_P)\big) \cap \pi_{F(Q)}\big(\mathrm{Gal}(V_Q, \mathscr{M}_Q)\big) = \pi_{F(R)}\big(\mathrm{Gal}(V_R, \mathscr{M}_R)\big)$$

A chamber admitting $F(P)$ and $F(Q)$ as facets also admits $F(R)$ as a facet. This proves (1.31) for the objects.

Let $g \in \mathrm{Hom}_{\mathrm{Gal}}(A, B)$, and suppose that

$$g = \pi_{F(P)}(g'_1) = \pi_{F(Q)}(g'_2).$$

From (1.27)(ii), we have

$$g_1' = \Delta(P)^{-n} g_1'' \quad \text{and} \quad g_2' = \Delta(Q)^{-n} g_2''$$

with g_i'' positive, for $n \geqq 0$ large enough. Let $g_1 = \pi_{F(P)} g_1''$ and $g_2 = \pi_{F(Q)} g_2''$.

From (1.26)(ii) (transformed by $*$), $\Delta^n \Delta(P)^{-n}$ is positive; in $\mathrm{Gal}_+(V, \mathscr{M})$ we have

(1.31.1) $$\left(\Delta^n \Delta(P)^{-n}\right) g_1 = \left(\Delta^n \Delta(Q)^{-n}\right) g_2.$$

(1.31.2) **Lemma.** *If $h \in \mathrm{Hom}_{\mathrm{Gal}_+}(A, B)$ starts with $\left(\Delta^n \Delta(P)^{-n}\right)$ or $\left(\Delta^n \Delta(Q)^{-n}\right)$, then h begins with $\left(\Delta^n \Delta(R)^{-n}\right)$.*

We deduce (1.31) from (1.31.2). From (1.31.2), we have

$$\left(\Delta^n \Delta(P)^{-n}\right) g_1 = \left(\Delta^n \Delta(Q)^{-n}\right) g_2 = \left(\Delta^n \Delta(R)^{-n}\right) g_3$$

with g_3 positive. We therefore have $g = \Delta(R)^{-n} g_3$. The positive gallery $g_3 = \Delta(R)^n g$ belongs to the image of $\pi_{F(P)}$ and of $\pi_{F(Q)}$. Thus (1.28) it crosses each wall which doesn't contain P or Q, i.e. doesn't contain R, zero times. It therefore belongs to the image of $\pi_{F(R)}$.

We now prove (1.31.2). Let $A(P) = A.\Delta.\Delta(P)$, and for each of Q and R. The gallery $\Delta\Delta(P)^{-1}$ when starting at A is $u(A, A(P))$; starting at $A(P)$ it is $u(A(P), A)$. From (1.26)(i) it follows that $\Delta\Delta(P) = \Delta(P)\Delta$. We then have

(1.31.3)
$$\Delta^n \Delta(P)^{-n} = u(A, A(P)).u(A(P), A)\, u(A, A(P)) \dots \quad (n \text{ factors}).$$

(1.31.4) **Lemma.** $A(R) \in D(A(P), A(Q))$.

With the notation of (1.2), $\mathscr{M}(A\Delta, A(P))$ is the set of walls which contain P and $\mathscr{M}(A\Delta, A(Q))$ is the set of walls which contain Q. From (1.2), $\mathscr{M}(A(P), A(Q))$ is the set of walls which contain P or Q, but not R, that is to say $\mathscr{M}(A(P), A(R)) \cup \mathscr{M}(A(Q), A(R))$, and we apply (1.4).

By (1.19)(iii), a gallery class which starts with $\Delta\Delta(P)^{-1}$ ($n > 0$) and $\Delta\Delta(Q)^{-1}$ thus also starts with $\Delta\Delta(R)^{-1}$, and (1.31.2) follows by induction on

(1.31.5) **Lemma.** *Let P and R be intersections of walls of a chamber A, with $P \subset R$. Let h be in* Gal_+ *and start with A. If h starts with both $\Delta^n \Delta(P)^{-n}$ ($n > 0$) and with $\Delta\Delta(R)^{-1}$, then h starts with $\left(\Delta\Delta(R)^{-1}\right)\left(\Delta\Delta(P)^{-1}\right)^{n-1}$.*

We can suppose that $n \geqq 2$. Set $h = \left(\Delta\Delta(P)^{-1}\right) h'$. With the previous notation, h' also starts with both $\Delta\Delta(P)^{-1} = u(A(P), A)$ and with $u(A(P), A(R))$. We apply (1.19)(iii).

The chamber C of loc. cit.[?] is also separated from $A(P)$ by each wall M which separates $A(P)$ from A (i.e. $M \not\supset P$) and $A(P)$ from $A(R)$ (i.e. $M \supset P$

and $M \not\supset R$). The chamber C is thus not separated from $A(P)\Delta$ by the walls $M \supset R$, and $C \in D\big(A(P).\Delta, A(P).\Delta\Delta(R)\big)$. Since then, h' starts both with $\big(\Delta\Delta(R)^{-1}\big)$ and with $\big(\Delta\Delta(P)^{-1}\big)^{n-1}$. Proceeding by induction on n, we then deduce that h' starts with $\big(\Delta\Delta(R)^{-1}\big)\big(\Delta\Delta(P)^{-1}\big)^{n-2}$, so that h starts with $\big(\Delta\Delta(P)^{-1}\big)\big(\Delta\Delta(R)^{-1}\big)\big(\Delta\Delta(P)^{-1}\big)^{n-2}$ and we conclude by noting

$$\Delta\Delta(P)^{-1}\Delta\Delta(R)^{-1} = \Delta\Delta(R)^{-1}\Delta\Delta(P)^{-1}.$$

(1.32) **Proposition.** *For a facet F of a chamber C, generating an intersection of walls P, we have*

$$\pi_F^{-1}\big(\mathrm{Gal}_+(V, \mathscr{M})\big) = \mathrm{Gal}_+(V_P, \mathscr{M}_P) \subset \mathrm{Gal}(V_P, \mathscr{M}_P).$$

Suppose that $\pi_F(\Delta^{-n} g) = h$ is in $\mathrm{Gal}_+(V, \mathscr{M})$. We then have $\pi_F(g) = \Delta(P)^n h$ and, by (1.28), each gallery H representing h does not cross the walls containing F. We thus have $h = \pi_F(h_1)$ with h_1 in $\mathrm{Gal}_+(V_P, \mathscr{M}_P)$ and we conclude by (1.30).

§ 2. Buildings

(2.1) Let $(V, \mathscr{M})$ be as in §1 $\big($satisfying (1.8)$\big)$ and $r = \dim V$. Let S be the sphere of radius one in V, for any Euclidean structure on V (one could more intrinsically put $S = V - \{0\}/\mathbb{R}_+^*$). The hyperplanes $M \in \mathscr{M}$ cut a triangulation of S, and we will still use S to denote the corresponding simplicial space and its geometric realization. We will carry over the terminology used for V (chambers, facets, adjacency, galleries, ...).

(2.2) Choose a "fundamental chamber" A_0 in S. We denote by I_+ the space constructed below; it depends on V, $\mathscr{M}$, and A_0, and is equipped with $q: I_+ \to S$.

a) Let $\mathscr{X}_+$ be the set of equivalence classes of galleries starting at A_0,

$$\mathscr{X}_+ = \coprod_B \mathrm{Hom}_{\mathrm{Gal}_+}(A_0, B).$$

b) Let Z be the disjoint union, indexed by $g \in \mathscr{X}_+$, of the closed simplices of S corresponding to the open simplices of dimension $r-1$ at which g ends,

$$Z_+ = \coprod_{g \in \mathscr{X}_+} (\text{end of } g)^-.$$

Let q' be the obvious map from Z_+ onto S.

c) I_+ (endowed with q) is a quotient of Z_+ (endowed with q'). If G is a gallery from A_0 to B and C is a chamber adjacent to B, we glue (end of g)$^-$ and (end of $g(BC)$)$^-$ along the common closed face of their images in S.

(2.3) The space I_+ is decomposed into facets, images of facets of Z_+ and bijectively mapped with a facet of S. Its *chambers* (resp. *faces*, resp. *vertices*) are the facets of dimension $r-1$ (resp. $r-2$, resp. 0).

The *vertices* of a facet F are the vertices contained in the closure $\overline{F}$ of F. The chambers of I^+ are indexed by $\mathscr{L}_+$; we let $\tilde{A}_0.g$ denote the chamber indexed by g and $\tilde{A}_0$ the chamber indexed by (A_0).

Let $g \in \mathrm{Hom}_{\mathrm{Gal}_+}(A_0, A)$, $\tilde{A} = \tilde{A}_0.g$, and $H = (C_0, \ldots, C_n)$ be a gallery from A to B of class h.

We set

$$\tilde{A}.h = \tilde{A}_0.gh.$$

The chambers $\tilde{A}.(C_0, \ldots, C_i)$ $(0 \leqq i \leqq n)$ form a gallery in I_+. We will call the galleries obtained thusly *positive*.

(2.4) **Lemma.** *Let F be a facet of S, A and B two chambers of S with F as a facet, $g \in \mathrm{Hom}_{\mathrm{Gal}_+}(A_0, A)$, and $h = \mathrm{Hom}_{\mathrm{Gal}_+}(A_0, B)$. The following conditions are equivalent*

(a) *in I_+, the facets $q^{-1}(F)$ of $\tilde{A}_0.g$ and $\tilde{A}_0.h$ coincide;*

(b) *$g^{-1}h \in \mathrm{Hom}_{\mathrm{Gal}}(A, B)$ is in $\pi_F(\mathrm{Gal})$;*

(c) *there exists g_1 and h_1 in $\pi_F(\mathrm{Gal}_+)$ such that $gg_1 = hh_1$.*

The condition (b) is an equivalence relation between galleries of A_0 starting at a chamber which has F as a facet. It follows that (a)⇒(b). That (b)⇒(c) follows from (1.26)(i). Finally, (c)⇒(a) follows trivially from the definitions.

From (2.4) (a)⇔(b) and (1.31), we immediately deduce the following.

(2.5) **Proposition.** *Each facet of I_+ is uniquely determined by the set of its vertices. We can therefore write I_+ as the geometric realization of the following simplicial complex.*

a) *For a vertex x of S, let $\mathscr{G}(x)$ be the set of g in* Gal_+ *from A_0 to a chamber with x as a vertex. Let $\mathscr{G}(x)/\pi_x$ be the quotient of $\mathscr{G}(x)$ by the equivalence relation* (2.4)(b) *(for $F = x$). Then, the set of vertices is*

$$\coprod_x \mathscr{G}(x)/\pi_x.$$

b) *For a set E of vertices to span a simplex, it is necessary and sufficient that there exist a gallery g starting at A_0 such that for $y \in E$, g is in $\mathscr{G}(qy)$ and y is the image of g in $\mathscr{G}(qy)/\pi_{qy}$.*

(2.6) **Proposition.** *Let F be a facet of I_+. There exists a class of galleries g such that, in order for F to be a facet of a chamber $B = \tilde{A}_0.h$, it is necessary and sufficient that we have*

$$h = g\pi_F(h')$$

for suitable h' in Gal_+.

Let us take g to be a gallery of minimal length such that F is a facet of $A = \tilde{A}_0.g$. Let $B = \tilde{A}_0.h$ be a chamber with F as a facet. Following (2.4)(a)⇔(c), there exist galleries h_1 and h_2 in $(V_{qF}, \mathscr{M}_{qF})$, with

$$g\pi_F(h_1) = h\pi_F(h_2).$$

Let us apply the transformation $*$ of (1.24). Given the minimality of g, we find that h_1 ends with h_2: $h_1 = h'h_2$. Following (1.19)(ii), we have

$$g\pi_F(h') = h$$

and this proves (2.6).

(2.7) **Notation.** Let A be a chamber of I_+. We let $S(A)$ denote the union of the closed chambers $\big(A.(qA, B)\big)^-$ for B a chamber of S.

One verifies that $q|S(A)$ is an isomorphism between $S(A)$ and S.

(2.8) **Definition.** $\hat{I}_+$ is the space derived from I_+ by ”filling” the spheres $S(A)$.

To be precise, let B^r be a ball whose boundary is S. When we want to work simplicially, we will take B^r to be the cone on the simplicial space S. We define $\hat{I}_+$ from I_+ by attaching a family of copies $b(A)$ of B^r, the family indexed by the set of chambers of I_+. The attaching maps are

$$\partial b(A) \xrightarrow{\sim} S \xleftarrow[\sim]{q} S(A).$$

The attaching maps are simplicial, so that $\hat{I}_+$ appears again like the geometric realization of a simplicial complex (whose vertices are those of I_+ and the “centers of the balls”). The space $\hat{I}_+$ is equipped with an obvious (simplicial) map

$$q : \hat{I}_+ \to B^r.$$

(2.9) **Proposition.** *The space $\hat{I}_+$ is contractible.*

The space I_+ is therefore a bouquet of the spheres $S(A)$.

Let $(I_+)_n$ be the union of the closed chambers $(\tilde{A}_0.g)^-$ for g of length $\leqq n$, and $(\hat{I}_+)_n$ the union of $(I_+)_n$ and the set of balls $b(A)$ for $\partial b(A) \subset (I_+)_n$.

We have $(\hat{I}_+)_0 = A_0^-$ (contractible), and

$$\hat{I}_+ = \varinjlim (\hat{I}_+)_n.$$

The proposition follows from

(2.9.1) **Lemma.** *$(\hat{I}_+)_n$ is a deformation retract of $(\hat{I}_+)_{n+1}$.*

We must construct a continuous family $(\varphi_t)_{0 \leqq t \leqq 1}$ of continuous maps of $(\hat{I}_+)_{n+1}$ to itself, with $\varphi_0 = \mathrm{Id}$, $\varphi_t | (\hat{I}_+)_n = \mathrm{Id}$, and $\varphi_1\big((\hat{I}_+)_{n+1}\big) = (\hat{I}_+)_n$.

Let $A = A_0.g$, a chamber of $(I_+)_{n+1}$ which is not in $(I_+)_n$, and let F be a facet of A.

(2.9.2) **Lemma.** *Suppose there is a chamber $B = \tilde{A}_0.h$ in $(I_+)_{n+1}$, distinct from A, of which F is a facet. There exists another chamber $B' = \tilde{A}_0.h'$ in $(I_+)_n$, and a wall M of qB', containing qF, such that $A = B'.\varDelta(M)$.*

Let g_0 be the gallery considered in (2.6). We have

$$g = g_0\, \pi_F(g_1) \quad \text{ and } \quad h = g_0\, \pi_F(h_1).$$

The hypotheses imply that the length $\lg(g_1) \neq 0$ (otherwise, since $A \neq B$, we would have $\lg(h) > \lg(g) = n+1$). For a suitable wall M of qA, we also have $g_1 = h'\varDelta(M)$, and we let $B' = \tilde{A}_0 h'$.

For the closed chamber A^-, $A^- \cap (I_+)_n$ is thus a union of a (nonempty) set of closed faces, and, in $(I_+)_{n+1}$, the points of $A^- - \big(A^- \cap (I_+)_n\big)$ belong solely to the closed chamber A^-.

We break into two cases

Case 1. $A^- \cap (I_+)_n \neq \partial A^-$.

In this case, there is no sphere $S(B)$ with

$$A \subset S(B) \subset (I_+)_{n+1}.$$

For $\varphi_t | A^-$, we take a collapsing of A^- onto $A^- \cap (I_+)_n$.

Case 2. $A^- \cap (I_+)_n = \partial A^-$.

Let $A = \tilde{A}_0.g$. For each wall M of qA, the result that g ends with $\varDelta(M)$ follows from (2.7.2).

According to $\big((1.19)(\mathrm{iii})\big)$, g therefore ends with $\varDelta$: we have $A = B.\varDelta$ with B uniquely determined by A, following (1.19)(ii). When passing from $(\hat{I}_+)_{n+1}$ to $(\hat{I}_+)_n$, A and the interior of the ball $b(B)$ disappear. For $\varphi_t | b(B)$, we take a collapsing of $b(B)$ onto $S(B) - A$.

From what was seen in case 1, any ball that disappears is of the previous type. This completes the construction of φ_t and proves (2.7).

(2.10) *The space I*, endowed with $q : I \to S$ is defined as I_+, replacing Gal_+ with Gal.

a) We set $\mathscr{Z} = \coprod_B \mathrm{Hom}_{\mathrm{Gal}}(A_0, B)$.

b) We set $Z = \coprod_{g \in \mathscr{Z}} (\text{end of } g)^-$.

c) Let q' be the obvious map from Z onto S. The space I, endowed with q, is a quotient of Z endowed with q'. For $g \in \mathscr{Z}$ with end B and C a chamber adjacent to B, we glue (end of g)$^-$ and (end of $g(BC)$)$^-$ along the common face of their closure of their images in S. In other words, if two chambers B and C have a common facet G, if $g \in \mathrm{Hom}_{\mathrm{Gal}}(A_0, B)$ and if

$h \in \mathrm{Hom}_{\mathrm{Gal}}(A_0, C)$, then, in I, the closed facets $q'^{-1}(F)^-$ of (end of g)$^-$ and of (end of h)$^-$ are identified if and only if

$$hg^{-1} \in \pi_F \mathrm{Hom}_{\mathrm{Gal}}\big(\pi_F^{-1}(B), \pi_F^{-1}(C)\big).$$

(2.11) As for I_+, we define the open or closed *chambers, faces, facets* of I. We define $\tilde{A}_0$, and $\tilde{A}.h$ (for $\tilde{A}$ a chamber and h in Gal starting at $q\tilde{A}$) as in (2.3). The analogue of (2.4)(a)⇔(b) is trivially true here. As in (2.5), we then infer from (1.31) that each facet of I is uniquely determined by its set of vertices, and I appears as the geometric realization of a simplicial complex which admits a description of type (2.5). For each chamber A of I, we define as in (2.7) a sphere $S(A)$; q induces an isomorphism of $S(A)$ onto S. As in (2.8) we define a building $\hat{I}$, equipped with $q: \hat{I} \to B^r$, by "filling" all the spheres $S(A)$.

(2.12) Besides q and its decomposition into chambers, I is equipped with the following additional structure:

– A "composition" $\tilde{B}.g$ consisting of a chamber $\tilde{B}$ of I with image B in S and $g \in \mathrm{Hom}_{\mathrm{Gal}}(B, C)$ associating a chamber of I with image C in S. The construction $g \mapsto \tilde{B}.g$ defines an isomorphism of the space analogous to I obtained by taking B as a fundamental chamber with I (for $A_0 = B$, this is an automorphism of I). In particular,

$$\tilde{B}.(gh) = (\tilde{B}.g).h.$$

(2.13) For each chamber A of I above A_0, we define a map

$$i_A: \quad I_+ \to I \quad (\text{resp. } \hat{I}_+ \to \hat{I}),$$

compatible with the projection onto S (resp. B^r) by sending the closed chamber $(A.g)^-$ of I_+ (resp. the ball $b(A.g)$ of $\hat{I}_+$) onto the chamber (resp. the ball) of the same name in I (resp. $\hat{I}$).

(2.14) **Proposition.** (i) *i_A identifies I_+ with a closed subspace of I and $\hat{I}_+$ with a closed subspace of $\hat{I}$.*

(ii) *We have*

$$I = \varinjlim_n i_{\tilde{A}_0.\Delta^{-2n}}(I_+) \quad \textit{and} \quad \hat{I} = \varinjlim_n i_{\tilde{A}_0.\Delta^{-2n}}(\hat{I}_+)$$

The assertions (i) and (ii) for $\hat{I}_+$ and $\hat{I}$ follow from the assertions (i) and (ii) for I_+ and I.

We prove (i) for I_+ and I. Let B and C be two chambers of S with a common facet F. Let $g \in \mathrm{Hom}_{\mathrm{Gal}_+}(A_0, B)$ and $h \in \mathrm{Hom}_{\mathrm{Gal}_+}(A, C)$. Suppose that the facets $q^{-1}F$ of $(A.g)^-$ and $(A.h)^-$ are equal in I.

We then have

$$g^{-1}h = \pi_F(e) \quad \text{for} \quad e \in \mathrm{Hom}_{\mathrm{Gal}}(\pi_F^{-1}B, \pi_F^{-1}C).$$

By (1.26), we can write $e = e_1 e_2^{-1}$ with e_i in Gal_+. By (1.29), we also have $g\pi_F(e_1) = g\pi_F(e_2)$ in Gal_+, and the facets $q^{-1}F$ of $(A.g)^-$ and $(A.h)^-$ are therefore also equal in I_+.

To prove (ii), it suffices to notice that any gallery g starting at A_0 in Gal can be written as $g = \Delta^{-2n}g'$ with g' in Gal_+

From (2.14) we find that $\pi_i(\hat{I}) = \varinjlim \pi_i(\hat{I}_+)$. Since I is a CW-complex, we deduce from (2.9) the following result.

(2.15) **Theorem.** *The space $\hat{I}$ is contractible.*

The space I is thus the bouquet of spheres $S(A)$ for A traversing over the chambers of I.

§ 3. Coverings

(3.1) Let $V_{\mathbb{C}}$ be the complexification of V, $M_{\mathbb{C}}$ the complexiciation of M for $M \in \mathscr{M}$, and

$$Y = V_{\mathbb{C}} - \bigcup_{M \in \mathscr{M}} M_{\mathbb{C}}.$$

In this paragraph, we construct a space $\tilde{Y}$ above Y via gluing. The gluing data will be described using I and $\hat{I}$. The facets of S and the facets of V other than $\{0\}$ are in canonical bijection, and we will constantly pass between them. In general, we will also note the corresponding facets; if necessary, the correspondence will be denoted $F \mapsto [F]$.

(3.2) **Lemma.** *Let A, B, and C be three chambers of I, with $C \subset S(A) \cap S(B)$. Then, q induces an isomorphism of $S(A) \cap S(B)$ with the intersection of S and the closed half-spaces $D'_M(qC)$ containing qC limited by a wall M which separates qA and qB.*

The intersection $D'_M(qC)$ is a union of closed chambers. If C' is one of them, qA and qB are on the same side of any wall which separates qC from C'. We deduce that the lifts to $S(A)$ or $S(B)$ of a minimal gallery from qC to C' coincide, and $(C')^- \subset q(S(A) \cap S(B))$.

Conversely, if a facet F is not in the intersection $D'_M(qC)$, there exists a wall M such that $F \not\subset M$ which separates qA from qB and F from qC. Suppose that it is impossible that $F \subset q(S(A) \cap S(B))$. Let D be a chamber of S with F as a facet and $(C_0, C_1 \ldots C_{n+1})$ be a gallery from qC to D. Let M_i be the wall which separates C_i and C_{i+1} and α_i (resp. β_i)= ± 1 depending on if C_{i+1} and A (resp. B) are or aren't on the same side of M_i. By assumption, F is a facet of $C\Delta(M_0)^{\alpha_0} \ldots C\Delta(M_n)^{\alpha_n}$ and of $C\Delta(M_0)^{\beta_0} \ldots C\Delta(M_n)^{\beta_n}$.

By (2.4) and (1.24), we then have (since $M \not\supset F$)

$$\sum_{M_i=M} \alpha_i = \sum_{M_i=M} \beta_i,$$

which is absurd: one side is worth one, the other -1.

(3.3) **Lemma.** *Let $\mathscr{R}$ and $\mathscr{I}$ be the "real part" and "imaginary part" mappings of $V_{\mathbb{C}}$ to V. Let $v \in V_{\mathbb{C}}$ and F be the facet of V such that $\mathscr{R}v \in F$. To have $v \in Y$, it is necessary and sufficient that* $\mathrm{pr}_F(\mathscr{I}v)$ *be in a chamber of* $(V_F, \mathscr{M}_F)$.

This is clear.

What follows is intuitively based on the fact that any "walk" in I (accomplished by following a gallery of the form $\Delta(M_0)^{\varepsilon_0} \dots \Delta(M_n)^{\varepsilon_n}$, $\varepsilon_i = \pm 1$) corresponds to a path ch (a class of paths) in Y. The image under $\mathscr{R}$ of ch in V passes from chamber to chamber, crossing successive walls $M_1, \dots, M_k$ at points m_i. For $\varepsilon_i = 1$ (resp -1) the path passes through one of two components R of $\mathscr{R}^{-1}(m_i) \cap Y$ (in bijection via $\mathscr{I}$ with $V - M$): this is such that $\mathscr{I}R$ contains (resp doesn't contain) the previous chamber.

(3.4) **Notation.** (i) Let C be a chamber of $(V, \mathscr{M})$. We let $Y(C)$ denote the open set of Y made up of the $x \in V_{\mathbb{C}}$ satisfying the following condition:

– if F is the facet of V such that $\mathscr{R}x \in F$, then $\mathrm{pr}_F \mathscr{I}x \in \pi'_F(C)$.

(ii) Let A and B be two chambers of I. If $S(A) \cap S(B)$ contains a chamber C, we temporarily let $Y'(A, B)$ denote the intersection of the open half spaces $D'_M(qC)^0$, for M as in 3.2. We set

$$Y(A, B) = Y(qA) \cap Y(qB) \cap \mathscr{R}^{-1} Y'(A, B) \subset V_{\mathbb{C}}.$$

If $S(A) \cap S(B)$ does not contain a chamber, we set $Y(A, B) = \emptyset$.

(iii) For a facet F of V, the *star* $Et(F)$ is the union of the (open) facets G of $(V, \mathscr{M})$ such that $F \subset G^-$.

(iv) For a facet F of V and a chamber C of $(V_F, \mathscr{M}_F)$, we set

$$V(F, C) = \{v \in V_{\mathbb{C}} | \mathscr{R}v \in Et(F) \quad \text{and} \quad \mathrm{pr}_F \mathscr{I}v \in C\}.$$

(3.5) **Lemma.** *Let G be a facet of V. The sets $V(F, C)$ for a facet F such that $G \subset F^-$ and a chamber C of V_F form an open covering of $Y \cap \mathscr{R}^{-1}(EtG)$. In particular (when $G = \{0\}$), the sets $V(F, C)$ cover Y.*

If $x \in Y \cap \mathscr{R}^{-1}(EtG)$ and $\mathscr{R}x \in F$, then x is in one of the sets $V(F, C)$ (3.3).

(3.6) Let Y' be the disjoint sum, indexed by the balls $b(A)$ of $\hat{I}$,

$$Y' = \coprod_{b(A)} Y(qA).$$

We let $Y'(b(A))$ denote the component of index $b(A)$. We then have an obvious projection $p' : Y' \to Y$. Let R be the following relation on Y':

For $x \in Y'(b(A))$ and $y \in Y'(b(B))$, $R(x,y)$ means that $p'(x) = p'(y)$ and that $p'(x), p'(y) \in Y(A,B)$.

This is an equivalence relation because (cf. (3.2)) $Y(A,B) \cap Y(B,C) \subset Y(A,C)$. We have

(3.6.1) The equivalence class under R of an open set of Y' is again open.

We set $\tilde{Y} = Y'/R$. We then have an obvious projection

$$p : \tilde{Y} \to Y.$$

For each ball $b(A)$ of $\hat{I}$, we let $\tilde{Y}(b(A))$ denote the image of $Y'(b(A))$ in $\tilde{Y}$.

The following theorem implies the first theorem of the introduction.

(3.7) **Theorem.** *$\tilde{Y}$ is a contractible covering of Y.*

A. *$\tilde{Y}$ is a covering of Y.*

We show, for $V(F,C)$ as in (3.4)(iv), that $p^{-1}V(F,C)$ is a sum of copies of $V(F,C)$.

a) Let $b(A)$ be a ball in $\hat{I}$. For a chamber B such that $F \subset B^-$, take the chamber of I

$$C'_B = A.u(A,B)\, u\big(\pi_F(C), B\big)^{-1}.$$

By construction, $B \subset q\big(S(A) \cap S(C'_B)\big)$ and in particular, if $F \neq \{0\}$,

$$[F] \subset q\big(S(A) \cap S(C'_B)\big). \tag{3.7.1}$$

We now prove that

$$\tilde{Y}\big(b(A)\big) \cap p^{-1}V(F,C) \subset \bigcup_{F \subset B^-} \tilde{Y}\big(b(C'_B)\big). \tag{3.7.2}$$

Let x be an element of the left hand side. Let G be the facet of V to which $\mathscr{R}\, p\, x$ belongs. We have $F \subset G^-$. For B, take a chamber which has G as a facet. By assumption, we have

$$\mathrm{pr}_G \mathscr{I}\, p\, x \in \pi'_G(A) \quad \text{and} \quad \mathrm{pr}_F \mathscr{I}\, p\, x \in C.$$

So, we have $\pi'_G(A) = \pi'_G(\pi_F C)$. We also deduce from (3.2) that each chamber containing G has a facet is in $q\big(S(A) \cap S(C'_B)\big)$, that $p\,x$ is in $Y(A, C'_B)$, and that therefore $x \in \tilde{Y}\big(b(C'_B)\big)$.

b) Let us put $Y'(F,C) = \bigcup_{q(C'_B)=C} \big(Y'(b(C')) \cap p'^{-1}(V(F,C))\big)$. By a), we have together

$$p^{-1}V(F,C) = Y'(F,C)/R.$$

This is also an isomorphism of topological spaces, by (3.6.1), because $Y'(F,C)$ is open in Y'. By (3.2), the equivalence relation induced by R on

$\bigcup_{q(C'_B)=C} Y'\big(b(C')\big)$ is trivial. We conclude by remarking that, for each C' above C,

$$V(F,C) \subset Y\big(b(C')\big).$$

B. *An open covering of $\tilde{Y}$.*

(3.7.3) **Notation.** (i) For F a facet of I,

$$\tilde{Y}(F) = \bigcup_{F \subset S(C)} \big(\tilde{Y}\big(b(C)\big) \cap p^{-1}\mathscr{R}^{-1}Et(qF)\big).$$

(ii) For $o(A)$ the center of a ball $b(A)$ of $\hat{I}$,

$$\tilde{Y}\big(o(A)\big) = \tilde{Y}\big(b(A)\big) \cap p^{-1}V(\{0\},A).$$

(3.7.4) **Lemma.** (i) *The sets $\tilde{Y}(s)$ for s a vertex of $\hat{I}$ form an open covering of $\tilde{Y}$ of which $\hat{I}$ is the nerve (i.e., a family of $\tilde{Y}(s)$ has a non-empty intersection if and only if the s correspond to the vertices of a facet of $\hat{I}$).*

(ii) *If a facet F of I has vertices $s_0, \ldots, s_p$, then*

$$\tilde{Y}(F) = \tilde{Y}(s_0) \cap \cdots \cap \tilde{Y}(s_p).$$

(iii) *If $F \subset S(A)$, then $\tilde{Y}\big(o(A)\big) \cap \tilde{Y}(F)$ is contractible; $\tilde{Y}(o(A))$ is contractible.*

(3.7.4.1) Let F_i $(i = 1,2)$ be two facets of I and C_i $(i = 1,2)$ two chambers of I such that $F_i \subset S(C_i)$ and that

$$\big(\tilde{Y}(b(C_1)) \cap p^{-1}\mathscr{R}^{-1}Et(qF_1)\big) \cap \big(\tilde{Y}(b(C_2)) \cap p^{-1}\mathscr{R}^{-1}Et(qF_2)\big) \neq \emptyset.$$

So, F_1 and F_2 span a facet F of I (unique by (2.5)–(2.11)), $F \subset S(C_i)$, and the former intersection coincides with

$$\tilde{Y}\big(b(C_1)\big) \cap \tilde{Y}\big(b(C_2)\big) \cap p^{-1}\mathscr{R}^{-1}Et(F).$$

By assumption

$$Y(C_1,C_2) \cap \mathscr{R}^{-1}Et(qF_1) \cap \mathscr{R}^{-1}Et(qF_2) \neq \emptyset;$$

F_1 and F_2 are in $S(C_1) \cap S(C_2)$ and they span a facet of F. Moreover, $Et(qF_1) \cap Et(qF_2) = Et(qF)$, hence the declared formula.

By (3.7.4.1), we deduce that $\tilde{Y}(F_1) \cap \tilde{Y}(F_2)$ is empty if F_1 and F_2 don't span a common facet F, and is equal to

$$\bigcap_{F \subset S(C)} \tilde{Y}(bC) \cap p^{-1}\mathscr{R}^{-1}Et(F) = \tilde{Y}(F)$$

otherwise (making $C_1 = C_2$). This shows (ii) and one part of (i).

(3.7.4.2) If $o(A) \neq o(B)$, then $\tilde{Y}(o(A)) \cap \tilde{Y}(o(B)) = \emptyset$.

If $qA = qB$, we have $\tilde{Y}(b(A)) \cap \tilde{Y}(b(B)) = \emptyset$. If $qA \neq qB$, we have

$$p\tilde{Y}(o(A)) \cap p\tilde{Y}(o(B)) = \emptyset.$$

(3.7.4.3) If $F \not\subset S(A)$, then $\tilde{Y}(o(A)) \cap \tilde{Y}(F) = \emptyset$.

If $F \subset S(B)$, we have $F \not\subset S(A) \cap S(B)$ and

$$Et(qF) \cap Y(A, B) = Et(qF) \cap p(\tilde{Y}(b(A)) \cap \tilde{Y}(b(B))) = \emptyset.$$

It is clear that:

(3.7.4.4) If $F \subset S(A)$, then p induces an isomorphism of $\tilde{Y}(o(A)) \cap \tilde{Y}(F)$ with $Y(qA) \cap \mathscr{R}^{-1} Et(qF)$.

Lemma (3.7.4) then results in:

(3.7.5) **Lemma.** *For each facet F of V and each chamber A,*

$$Y(A) \cap \mathscr{R}^{-1} Et(F)$$

is contractible.

Let v_0 be a vector such that $\mathscr{R} v_0 \in F$ and $\mathscr{I} v_0 \in A$. Then for $0 \leqq t \leqq 1$, $\varphi_t : v \mapsto v_0 + t(v - v_0)$ is a map from $Y(A)$ and $\mathscr{R}^{-1} Et(F)$ to themselves. We have $\varphi_0(Y(A) \cap \mathscr{R}^{-1} Et(F)) = \{v_0\}$ and $\varphi_1 = \mathrm{Id}$.

C. *Study of* $\tilde{Y}(F)$ (F a facet of I).

By (3.5), (3.7.1), and (3.7.2), we have

$$\tilde{Y}(F) = \bigcup_{F \subset B^-} \tilde{Y}(b(B)) \cap p^{-1} \mathscr{R}^{-1} Et(F). \tag{3.7.6}$$

We choose a "fundamental chamber" $A_{F,0}$ in $(V_F, \mathscr{M}_F)$ and let I_F denote the corresponding building. We also choose $\tilde{A}^I_{F,0}$ in I above $\pi_F(A_{F,0})$. For each chamber $B = \tilde{A}_0.g_F$ of I_F, we denote by $\pi_F(B)$ the chamber $\tilde{A}^I_{F,0}.\pi_F(g_F)$ of I. By (2.10), (3.7.6) can be rewritten as

$$\tilde{Y}(F) = \bigcup_{B \text{ in } I_F} \tilde{Y}(b(\pi_F(B))) \cap p^{-1} \mathscr{R}^{-1} Et(F).$$

Let

$$Y_F = (V_F)_{\mathbb{C}} - \bigcup_{M \in \mathscr{M}_F} M_{\mathbb{C}}$$

and $\tilde{Y}_F$ be the covering of Y_F defined using I_F.

Whenever B and C are in I_F, it follows from (1.30) that the chambers of S in $Y(\pi_F(B), \pi_F(C)) \cap Et(F)$ are exactly the $\pi_F(D)$ for D in $Y_F(B, C)$.

The gluings that define $\tilde{Y}(F)$ and $\tilde{Y}_F$ are also compatible, and there exists a unique commutative diagram

(3.7.7)
$$\begin{array}{ccc} \tilde{Y}_F & \longleftarrow & \tilde{Y}(F) \\ \downarrow & & \downarrow \\ Y_F & \xleftarrow{\mathrm{pr}_F} & \mathscr{R}^{-1}Et(F) \subset Y \end{array}$$

which sends $\tilde{Y}\big(b(\pi_F(B))\big) \cap p^{-1}\mathscr{R}^{-1}Et(F)$ into $\tilde{Y}_F\big(b(B)\big)$ for B a chamber of I_F. The first vertical arrow thus also the second are coverings.

One easily verifies that the map pr_F of (3.7.7) is a homotopy equivalence, from which

(3.8) **Lemma.** *$\tilde{Y}_F$ and $\tilde{Y}(F)$ have the same homotopy type.*

D. *End of the proof.*

We will prove (3.7) by induction on $r = \dim V$. Let $\mathscr{U}$ be the open covering of $\tilde{Y}$ by the $\tilde{Y}(s)$, for a vertex s of $\hat{I}$. By (3.7.4)(i), $\hat{U}$ is the nerve of $\mathscr{U}$. By (3.7.4), (3.8), and the induction hypothesis, the non-empty intersections of open sets belonging to $\mathscr{U}$ are contractible. This implies [6] that $\tilde{Y}$ and $\hat{I} = \mathrm{Nerve}(\mathscr{U})$ have the same homotopy type. We conclude with (2.15).

§ 4. Braid groups

(4.1) Let V be a finite dimensional real vector space and $W \subset GL(V)$ be a finite group generated by reflections. We suppose that $V^W = \{0\}$. Let Φ be any euclidean structure invariant under W and let $\mathscr{M}$ be the set of hyperplanes M such that the orthogonal reflection corresponding to $\mathscr{M}$ is in W. We know that

a) $(V, \mathscr{M})$ satisfies 1.6 ([1] V 3.9 prop. 7);

b) W permutes the chambers of $(V, \mathscr{M})$ strictly simply transitively ([1] V 3.2 Th. 1);

c) if $x \in V$ belongs to the closed chamber $\overline{A}$, its stabilizer is generated by the reflections with respect to the walls of A containing x ([1] V 3.3 prop. 1);

d) (following from c)) the group W acts freely on $Y_W = V_{\mathbb{C}} - \bigcup_{M \in \mathscr{M}} M_{\mathbb{C}}$.

From b), if C_0 and C_1 are two chambers, there exists a unique $w \in W$ such that $wC_0 = C_1$; this w induces a transitive system of bijections between the walls of the different chambers. Passing to the quotient, one obtains a set D of $\dim(V)$ elements and, for each chamber C, a bijection φ_C of D with the set of walls of C. We have

(4.1.1) $$\varphi_{wC}(i) = w\varphi_C(i) \quad (w \in W).$$

For the definition of the *Coxeter matrix* $(m_{ij})_{i,j\in D}$ and that of the *Coxeter graph*, we refer to [1] V 3.4.

(4.2) Choose a chamber A_0 of $(V, \mathscr{M})$. For $i \in D$, let w_i be the reflection based on the wall $\varphi_{A_0}(i)$. [1] V 3.2 Th. 1 says that W is generated by the w_i, these generators only being subject to the relations

$$(w_i w_j)^{m_{ij}} = e.$$

(4.3) Let $X_W = Y_W/W$: by (4.1)d), Y_W is a covering of X_W. Let $y_0 \in A_0$ have image $x_0 \in X_W$. For each $i \in D$, let ℓ_i' be the homotopy class of paths in Y_W, from y_0 to $w_i(y_0)$, which, for $y \in A_0$, contains the broken line of successive vertices y_0, $y_0 + iy_0$, $w_i(y_0) + iy_0$, $w_i(y_0)$. The image ℓ_i of ℓ_i' in X_W is a loop with basepoint x_0.

(4.4) **Theorem.** (i) *The fundamental group $\pi_1(X_W, x_0)$ is generated by the ℓ_i; these are subject only to the relations*

(4.4.1) $$\operatorname{prod}(m_{ij}; \ell_i, \ell_j) = \operatorname{prod}(m_{ji}; \ell_j, \ell_i).$$

(ii) *The universal cover of X_W is contractible: X_W is a $K(\pi; 1)$.*

Assertion (i) is proved in Brieskorn [3]. Another proof will be outlined in (4.18) below. As explained in the introduction, (ii) results from (3.7) and (4.1).

(4.5) In §1, we drew heavily on the study of the Artin braid groups by Garside [4]; from the results obtained, we can now deduce information on all the groups of type $\pi_1(X_W, x_0)$ (Garside had already considered some of them). The translation is based on (4.6), (4.7) below.

(4.6) Let $G = (C_0, \ldots, C_n)$ be a gallery and M_i the wall which separates C_{i-1} from C_i $(1 \leqq i \leqq n)$. We have $\varphi^{-1}_{C_{i-1}}(M_i) = \varphi^{-1}_{C_i}(M_i)$. We denote by $\ell(G)$ the element $\varphi^{-1}_{C_1}(M_1) \ldots \varphi^{-1}_{C_n}(M_n)$ of $L^+(D)$ (0.1).

The map ℓ induces a bijection of the set of galleries G of a given source in $L^+(D)$. We have

(4.6.1) $$\ell\big(w(G)\big) = \ell(G) \qquad \text{(for } w \in W\text{)};$$

(4.6.2) $$\ell(G_0\, G_1) = \ell(G_0)\, \ell(G_1) \qquad \text{(for } G_0 \text{ and } G_1 \text{ composable)}.$$

Let $w : L(D) \to W$ be the homomorphism which extends the map $i \mapsto w_i$ from D onto W.

(4.7) **Proposition.** *For each gallery G of source A_0, we have*

$$A_0.G = w\big(\ell(G)\big)(A_0).$$

This is an evolution of the identity

$$[(w_1 \dots w_{n-1})\, w_n (w_1 \dots w_{n-1})^{-1}] \cdot [(w_1 \dots w_{n-2})\, w_{n-1} (w_1 \dots w_{n-2})^{-1}] \dots [w_1] = w_1 \dots w_n .$$

(4.8) From this proposition we deduce that, for $w \in W$, the integer $d(A_0, w(A_0))$ is the smallest length of words in w_i which equal w; more precisely, ℓ induces a bijection of the set of minimal galleries from A_0 to wA_0 with the set of words $m \in L(D)$ of minimal length such that $w = w(m)$.

(4.9) **Definitions.** (i) *The braid monoid G^+ of type D is the unitial monoid generated by D, the generators $i \in D$ being subject only to the relations*

(4.9.1) $$\mathrm{prod}(m_{ij}; i, j) = \mathrm{prod}(m_{ij}; j, i)$$

(ii) *The braid group G (or generalized braid group) of type D is the group generated by D, the generators being subject only to* (4.9.1).

(4.10) We note that W again admits the presentation

(4.10.1) $$\mathrm{prod}(m_{ij}; w_i, w_j) = \mathrm{prod}(m_{ij}; w_j, w_i)$$

(4.10.2) $$w_i^2 = e.$$

so that $i \mapsto w_i$ extends to a homomorphism of G onto W.

(4.11) One may verify by the definitions that two galleries G_0 and G_1 of the same source are equivalent if and only if $\ell(G_0)$ and $\ell(G_1)$ have the same image in G^+. Taking into account (4.9), (1.12) therefore admits the following adaptation ([1] IV 1.6 prop. 5).

(4.12) **Recall.** *Let $w \in W$. The image of the words $m \in L(D)$ in G^+ of minimal length among those such that $w = w(m)$ depends only on w.*

This image will be denoted $r(w)$; r is a section of $w : G^+ \to W$. By (4.7), we have $r(w) = \ell\big(u(A_0, wA_0)\big)$.

Likewise, [1] IV 1.4 lemma 2 corresponds to (1.11).

(4.13) The "quotient" category of $\mathrm{Gal}_+(V, \mathscr{M})$ by W is defined as follows:

a) The set of objects is reduced to one element: this is

$$\mathrm{Ob}\big(\mathrm{Gal}_+(V, \mathscr{M})\big)/W.$$

b) The monoid of arrows is $\mathrm{Ar}\big(\mathrm{Gal}_+(V, \mathscr{M})\big)/W$, and passing to the quotient

$$\mathrm{Gal}_+(V\mathscr{M}) \to \mathrm{Gal}_+(V,\mathscr{M})/W$$

is a functor.

This category, having only one object, is identified with the monoid of its arrows. By (4.6.1), (4.6.2), and (4.11), ℓ identifies this monoid with G^+.

We define $\mathrm{Gal}(V,\mathscr{M})/W$ analagously. This category has only one object and, by its universal property, ℓ extends to an isomorphism of the group of its arrows with G:

$$\begin{array}{ccccc}
\mathrm{Gal}_+(V,\mathscr{M}) & \longrightarrow & \mathrm{Gal}_+(V,\mathscr{M})/W & \xrightarrow[\ell]{\sim} & G^+ \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{Gal}(V,\mathscr{M}) & \longrightarrow & \mathrm{Gal}(V,\mathscr{M})/W & \xrightarrow[\ell]{\sim} & G
\end{array}$$

(4.14) **Theorem.** (i) *In G^+, the left and right translation maps are injective.*

(ii) *G^+ satisfies the left and right conditions of Öre. It follows from* (i) *that G^+ embeds in G.*

(iii) *For a subset J of D, let G^+_J (resp. G_J) be the submonoid (resp. subgroup) of G^+ (resp. G) generated by the $i \in J$. Then, the obvious map identifies G^+_J (resp. G_J) with the braid monoid (resp. braid group) of type J.*

(iv) *We have $G^+_J = G_J \cap G^+$.*

(v) *If J and K are two subsets of D, we have $G_J \cap G_K = G_{J\cap K}$.*

(vi) *Let $n(i)$ be the number of elements of G^+ of length i and set $f = \sum n(i)t^i$. For $J \subset D$, let $m(J)$ be the number of walls which pass through the intersection of walls of one (any) chamber A, indexed by J (= number of reflections in the Weyl group W_J). We have*

$$f = \Big(\sum_{J\subset D} (-1)^{|J|} t^{m(J)}\Big)^{-1}.$$

These assertions translate respectively: (i): (1.1)(i) and (ii); (ii): (1.26)(ii) and its transformation by passing to opposite galleries (cf. also (4.16) below); (iii): (1.30); (iv): (1.32); (v): (1.31); (vi): (1.21) (the f_A of *loc. cit.* are all equal to f).

Let $x, y \in G^+$. We say that x *starts* with y if there exists a z in G^+ with $x = yz$. The proposition (1.19)(iii) then translates as

(4.15) **Lemma.** *Let $x \in G^+$. There exists a chamber C such that, for each $w \in W$, x starts with $r(w)$ if and only if $wA_0 \in D(A_0, C)$.*

(4.16) Let J be a subset of D and P the intersection of the walls $\varphi_{A_0}(i)$ $(i \in J)$. We let $\Delta(J)$ denote the element $\ell(u(A_0, A_0.\Delta(P)))$ in G^+; we set $\Delta = \Delta(D)$. Let $i \mapsto \bar{i} = \varphi_{-C}^{-1}\varphi_C(i)$ be the *opposite involution* of D. We also let $w \mapsto \overline{w}$ denote the automorphisms of $L^+(D)$, G^+, and G which send i to $\bar{i}$ (we have $m_{ij} = m_{\overline{ij}}$). Adapting (1.26) with the aid of (4.4), we find that for g in G^+ or G, we have

(4.16.1)
$$g\Delta = \Delta\overline{g}.$$

In particular, Δ^2 is central.

For any $i \in D$, Δ "starts" with i: we have $\Delta = ix$, in G^+. Since then

(4.17) **Proposition.** *G is derived from G^+ by making the central element Δ^2 invertible.*

(4.18) The description (2.11), (2.5)–(2.7) of the building I associated with $(V, \mathscr{M})$ can be translated as well.

a) The set of vertices of I is

$$I^0 = \coprod_{i \in D} G/G_{D-\{i\}}.$$

b) For a set of vertices to span a simplex, it is necessary and sufficient that it be contained in the set of cosets $gG_{D-\{i\}}$, for some $g \in G$.

c) The fundamental sphere $S_0 = S(A_0)$ is the union of the fundamental simplex (spanned by the coset of e in $G/G_{D-\{i\}}$) and of its transforms by $r(w)$ $(w \in W)$.

The preceding description gives an obvious left action of G on I. It respects the set of spheres $S(A)$ and the correspondence $A \mapsto S(A)$.

By transporting the structure, G acts on the covering $\tilde{Y}_W$ of Y_W (3.6), we have an equivariant morphism

$$(G - \tilde{Y}_W) \to (W - Y_W).$$

We conclude that G is the fundamental group of $X_W = Y_W/W$, and (4.4)(i).

(4.19) To be complete, let us show that the word problem, the conjugacy problem, and the question of calculating the center of G can be resolved as in [4]. If the Coxeter graph D of W is the disjoint sum of graphs D_α, the braid group of type D is a product of the braid groups of type D_α. The essential case is therefore that of a connected graph D.

(4.20) *The word problem*: there is a decision procedure to know if two elements m and n of $L(D)$ have the same image in G.

a) It is known that it is decidable that m and n in $L^+(D)$ have the same image in G^+, because the imposed relations do not change the length of the words, and there are only a finite number of words of given length. One can also use (1.22).

b) For any m_i ($i = 1, 2$) in the free group $L(D)$ generated by D, we can calclate $m_i' \in L^+(D)$ and $k \geqq 0$ such that m_i and $m_i' \Delta^{-k}$ have the same image in G (cf. (4.17)); m_1 and m_2 then have the same image in G if and only if m_1' and m_2' have the same image in G^+.

(4.21) **Theorem.** *If the (non-empty) Coxeter graph D is connected and the opposite involution is trivial (resp. non-trivial), the center of G is infinite cyclic, generated by Δ (resp. Δ^2).*

Consider the following property of $x \in G^+$.

(∗) For any $i \in D$, there exists an $i' \in D$ such that $ix = xi'$.

It suffices to show that, if x satisfies (∗) and $x \neq e$, then $x = \Delta y$ with $y \in G^+$. By induction on the length of x, we will deduce that if x satisfies (∗), then x is of the form Δ^k ($k \geqq 0$); by (4.14)(i) and (4.17), such elements of G are contained in the set of Δ^k ($k \in \mathbb{Z}$) and the assertion follows from (4.16.1).

So, let $x \in G^+$ satisfy (∗). Let C be as in (4.15) and w the image of $\ell u(A_0, C)$ in W. We can write $x = r(w)y$, with $y \in G^+$. Let $i \in D$. By (∗), $ix = r(w)yi'$ starts with $r(w)$; from (1.23), we then conclude that $ir(w)$ starts with $r(w)$: $ir(w) = r(w)i''$. In particular, for each i, there exists an i'' with $iw = wi''$; this means that any wall of A_0 is also a wall of wA_0. Among the $2^{|D|}$ open simplicial cones bounded by the walls A_0, only A_0 and $-A_0$ are chambers: these are the only ones whose angles between the faces are all $\leqq \pi/2$ (here, we use the connectivity of D). If $x \neq e$, we have that $w \neq e$ and so $wA_0 = -A_0$. Therefore $\Delta = r(w)$, completing the demonstration.

(4.22) *The derived group.* Let D' be the graph with D as its set of vertices, two vertices being connected by an edge if and only if m_{ij} is odd. Let D_0 be the set of connected components of D'. We define an epimorphism

$$\mathrm{Lg} : G \to \mathbb{Z}^{D_0}$$

by sending i to the basis vector corresponding to the connected component of D_0 containing i. One shows immediately that Lg idenifies $\mathbb{Z}^{D_0}$ with the largest abelian quotient of G.

(4.23) *The conjugation problem.* This is a question of giving a decision process to know if two elements x and y of G (given explicitly as images of words in $L(D)$) are conjugates. Since Δ^2 is central, $\Delta^{-2k}a$ and $\Delta^{-2k}b$ are conjugate if and only if a and b are. By (4.17), it is therefore sufficient to treat the case where $x, y \in G^+$. Likewise, if x and y are conjugates, there is an a *in* G^+ such that $xa = ay$. If x and y are conjugates, $\mathrm{Lg}(x) = \mathrm{Lg}(y)$. There are only a finite number of $x \in G^+$ of given "length" $\mathrm{Lg}(x)$ and W is finite. The existence of a decision process then follows from (4.20) and the following lemma.

(4.24) **Lemma.** *Let x, y be in G^+. If x and y are conjugates, there exists a sequence $x_0 = x, x_1, \ldots, x_n = y$ of elements of G^+ and a sequence of elements w_i of W such that*

$$x_{i+1} r(w_i) = r(w_i) x_i .$$

Let $a \in G^+$ be such that $xa = ay$. Set $a = r(w_0) \ldots r(w_{n-1})$, with w_i the element of maximal length in W such that $r(w_i) \ldots r(w_{n-1})$ can be written in the form $r(w_i) b_i$ with $b_i \in G^+$ (4.15). Let $a_i = r(w_0) \ldots r(w_i)$. We prove that $x_i = a_i^{-1} x a_i$ is in G^+, so that the x_i and w_i will answer the problem.

It follows from (1.23) that, for all $w \in W$, if xa starts with $r(w)$, then $xr(w_0)$ also starts with $r(w)$. In particular, since $xa = ay = r(w_0) b_0 y$ starts with $r(w_0)$, $xr(w_0) r(w_0) x_1$ with $x_1 \in G^+$. We complete the proof by proceeding by induction on n.

Remark. Let σ be an automorphism of the Coxeter graph D, and let σ also denote the corresponding automorphism of G. We have $\Delta^\sigma = \Delta$. The preceding arguments still apply to the question of knowing, for $x, y \in G$, if there exists a in G such that $xa = a^\sigma y$. This reduces to taking x, y, and a in G^+. In this case and, with the previous notation, if $xa = a^\sigma y$, the $(a_i^{-1})^\sigma x a_i$ are in G^+. Taking σ to be the involution $i \mapsto \bar{i}$, we find a criterion analagous to (4.24) for the conjugation of Δx and Δy $(x, y \in G^+)$.

Bibliography

The reader will find a more complete bibliography on braid groups in [2].

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