CAT(0) AND CUBULATED SHEPHARD GROUPS

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ABSTRACT. Shephard groups are common generalizations of Coxeter groups, Artin groups, and graph products of cyclic groups. Their definition is similar to that of a Coxeter group, but generators may have arbitrary order rather than strictly order 2. We extend a well known result that Coxeter groups are CAT(0) to a class of Shephard groups that have "enough" finite parabolic subgroups. We also show that in this setting, if the associated Coxeter group is type (FC), then the Shephard group acts properly and cocompactly on a CAT(0) cube complex. As part of our proof of the former result, we introduce a new criteria for a complex made of A_3 simplices to be CAT(1).

1. INTRODUCTION

The classical notion of a *Shephard group* first arose in G. C. Shephard's thesis [She52]. In it, Shephard introduces the notion of a regular complex polytope and studies the symmetry groups of such an object, which have been termed Shephard groups after his work. It turns out that these symmetry groups are complex reflection groups (finite subgroups of $GL_n(\mathbb{C})$ generated by complex linear reflections) and have a "Coxeter-like" presentation. This group presentation can be encoded in a "Coxeter-like" diagram, with certain restrictions on labels and shape of the diagram. In this paper, we study those groups defined by presentations identical to those of the classical Shephard groups, but without these same restrictions on the diagrams. Their precise definition is as follows.

Let Γ be a simplicial graph with vertex set I and the following information. For each vertex $i \in I$, we assign a number $p_i \in \mathbb{Z}_{\geq 2} \cup \{\infty\}$. For each edge $\{i, j\}$ of Γ, we assign a number $m_{ij} = m_{ji} \in \mathbb{Z}_{\geq 3} \cup \{\infty\}$. If $\{i, j\}$ is not an edge, define $m_{ij} = 2$. If m_{ij} is odd, then we require that $p_i = p_j$. Sometimes if $p_i = 2$ or $m_{ij} = 3$ we omit the label. We say this data defines an *extended Coxeter diagram*. Associated to such a diagram Γ, we define the *Shephard group* G_Γ on generators $S = \{ s_i : i \in I \}$ with the following presentation:

$$G_\Gamma = \left\langle S \ \middle| \ \begin{matrix} \mathrm{prod}(s_i, s_j; m_{ij}) = \mathrm{prod}(s_j, s_i; m_{ij}) \\ s_i^{p_i} = 1 \end{matrix} \right\rangle, \tag{1.1}$$

where $\mathrm{prod}(a, b; m)$ denotes $(ab)^{m/2}$ if m is even and $(ab)^{(m-1)/2}a$ if m is odd. (When $p_i = \infty$ or $m_{ij} = \infty$, the corresponding relation is omitted.) This is why we require $p_i = p_j$ when m_{ij} is odd; s_i and s_j are conjugate, and thus if $p_i \neq p_j$ the order of s_i and s_j would not necessarily be p_i or p_j, respectively. Note that if Γ is disconnected, then G_Γ splits as a direct product over the connected components of Γ.

We point out some special cases to show the power of this general definition.

(1) If all $p_i = 2$, then G_Γ is a Coxeter group.
(2) If all $p_i = \infty$, then G_Γ is an Artin group.

(3) If all $m_{ij} = 2$ or ∞ (in which case we may call G_Γ "right-angled"), then G_Γ is a graph product of cyclic groups.

This is, however, not simply an empty generalization to compile the above groups into one succinct class. Many interesting infinite Shephard groups (and their quotients) appear in various places in mathematics. For example, they arise naturally in algebraic geometry as objects closely related to moduli spaces of arrangements in $\mathbb{CP}^2$ [KM98]. We detail further examples in the following.

Example 1.1. In certain cases, Shephard groups have (typically proper) quotients to infinite affine and hyperbolic complex reflection groups; a particularly interesting class of examples comes from the Shephard groups G_Γ with diagram Γ given in Figure 1.
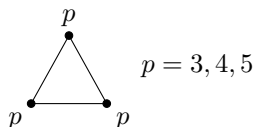

$p = 3, 4, 5$

FIGURE 1. A class of infinite Shephard groups

When $p = 3$, G_Γ has a quotient to one of Popov's *affine complex reflection groups* [Pop22]. It is demonstrated in [CM21] that this quotient (refered to as $\mathrm{REFL}(\widetilde{G_4})$ in said article) possesses a quite elegant geometry: the complement of its reflection hyperplanes deformation retracts to a non-positively curved 2-dimensional complex built out of $(3, 3, 3)$ triangles, whose link is a Mobius-Kantor graph. This link is isomorphic (and isometric) to the link of a complex we define for arbitrary Shephard groups—but this $(3, 3, 3)$ triangle complex for $\mathrm{REFL}(\widetilde{G_4})$ is *not* the same as our complex for G_Γ. (It may be interesting to further study the exact relationship between these two complexes.)

Further, when $p = 3, 4$ or 5, G_Γ has quotients to the hyperbolic complex reflection groups upon which Mostow's famous examples of complex hyperbolic polyhedra in $\mathbb{CH}^2$ and non-arithmetic lattices of $\mathrm{PU}(2, 1)$ are based [Mos80].

Another interesting pair of examples comes from the following diagram, which we call $B_n(p, q)$:
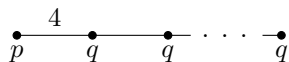
This family of Shephard groups finds use in the dual Garside theory of Artin groups. The dual Garside theory has shown to be quite effective in making progress on the $K(\pi, 1)$ conjecture (e.g., in the recent proof of the conjecture for the affine-type Artin groups [PS21]).

Example 1.2. In [BG17], it is shown that the Artin group of type D_n embeds into the Shephard group of type $B_n(2, \infty)$ as a finite-index subgroup. This is used to classify the so-called *Mikado braids* (certain well-behaved words) of the D_n Artin group, and specifically, to show that the set of simple elements under its dual Garside structure are all Mikado braids.

Example 1.3. In [MS17], the Shephard group of type $B_n(\infty, 2)$ is called the "middle group" $\text{MID}(B_n)$, and is defined explicitly as a certain semidirect product of translations and reflections in $\mathbb{R}^n$. This group plays a key technical role in the proofs of the facts that, for a general affine-type Artin group, it is isomorphic to its dual Artin group, and this dual Artin group embeds in a Garside group. (It is these results that form part of the foundation of [PS21].)

Much of our work relies on the close relationship of Shephard groups and Coxeter groups, and so we make the following definition. For any extended Coxeter diagram, we let W_Γ and A_Γ denote the Coxeter group and Artin group, resp., defined by the underlying Coxeter diagram of Γ (that is, the Coxeter diagram obtained from Γ by ignoring the vertex labels). We recall that these groups have the following presentations:

$$A_\Gamma = \langle\, S \mid \text{prod}(s_i, s_j; m_{ij}) = \text{prod}(s_j, s_i; m_{ij}) \,\rangle,$$
$$W_\Gamma = \langle\, S \mid (s_i s_j)^{m_{ij}} = 1, s_i^2 = 1 \,\rangle,$$

If the diagram Γ has all $p_i = 2$ (so $G_\Gamma = W_\Gamma$), we will sometimes call Γ "Coxeter" (without the "extended") and otherwise call Γ "non-Coxeter". But we emphasize that we declare W_Γ is a Coxeter group regardless of the vertex labeling of Γ.

1.1. **Description of results.** We now give a broad overview of the results of this paper.

We heavily use the fact that G_Γ is finite for certain diagrams Γ. If G_Γ is a finite group, we sometimes call the diagram Γ itself "finite". It is an interesting fact that there are finite (abstract) Shephard groups G_Γ which are not themselves Coxeter groups. See Table 4 for examples. The diagrams in this table come from the the well-known classification of finite complex reflection groups. However, it is a priori unclear if there is some diagram Γ, say, which is branched or a cycle, so that G_Γ is a finite group under this abstract presentation, but does not correspond to any complex reflection group. We confirm in Theorem 2.7 that this cannot happen; that is, the only finite abstract Shephard groups G_Γ are those arising from the complex reflection groups (specifically, those in Tables 3 and 4).

We are able to classify the finite abstract Shephard groups, but the infinite Shephard groups remain rather mysterious. It is unclear if there is a unified topological interpretation of the Shephard groups, compared to the relationship between Coxeter groups and real reflection groups, and that between Artin groups and hyperplane complements. One of the main objectives of this article is to propose a geometric model for the infinite Shephard groups which, in many cases, can play the role of the Davis complex for a given Coxeter group. Similarly to the Davis complex, this model is built out of the finite "parabolic" subgroups of the Shephard group. To make this more precise, we make the following definitions.

For an extended Coxeter diagram Γ, a *subdiagram* is a **full** subgraph Γ' of Γ inheriting the vertex and edge labels of Γ. (Recall that a subgraph is full if whenever two vertices of the subgraph are joined by an edge in the original graph, they remain joined by an edge in the subgraph.) If Γ' is a subdiagram of Γ, then $G_{\Gamma'}$ is also a Shephard group. The group $G_{\Gamma'}$ surjects onto the subgroup of G_Γ generated by the vertex set of Γ' viewed within Γ. To state this more precisely, and momentarily avoid the question of whether this is an isomorphism or not, we introduce the following notation: let $S = \{\, s_i : i \in I \,\}$ denote the generating set of G_Γ, and let $J \subseteq I$ with

$T = \{\, s_j : j \in J \,\} \subseteq S$. Then define $G_T = \langle T \rangle$ to be the subgroup of G_Γ generated by T, and define $G_{\Gamma(T)}$ to be the Shephard group defined by $\Gamma(T)$, the full subgraph of Γ on vertex set J.

We now describe the main complex on which the Shephard groups act. Consider the following posets.

$$\mathcal{S}_\Gamma^f = \mathcal{S}^f = \{\, T \subseteq S : W_{\Gamma(T)} \text{ is finite} \,\},$$
$$\mathcal{S}_\Gamma^{fs} = \mathcal{S}^{fs} = \{\, T \subseteq S : G_{\Gamma(T)} \text{ is finite} \,\}.$$

The set $\mathcal{S}^f$ plays a major role in the study of Coxeter groups as the basis of the fundamental domain of the Davis complex. Its direct analogue in a Shephard group is $\mathcal{S}^{fs}$. We continue the analogy with Coxeter groups to define a complex (in Definition 3.3), which we denote $\Theta = \Theta_\Gamma$, on which G_Γ acts properly and cocompactly.

It would be too much to hope that this complex behaves nicely for all Shephard groups; after all, Γ could have all vertex labels ∞, in which case $\mathcal{S}^{fs}$ would be empty. But with certain restrictions on the number and type of finite subgroups, we can show that Θ has a very nice geometry, and is, in fact, CAT(0). Our first step in this direction is the following.

Theorem A. *Suppose Γ is an extended Coxeter diagram with $\mathcal{S}_\Gamma^f = \mathcal{S}_\Gamma^{fs}$ such that Γ has no subdiagram of type "$A_4(3)$" (see Table 4). Then Θ_Γ is* CAT(0)*, and hence G_Γ is* CAT(0) *(i.e., acts properly and cocompactly on a* CAT(0) *space by isometries).*

Some examples of extended Coxeter diagrams satisfying the hypothesis of Theorem A can be found in Table 1. Among the other nice properties of CAT(0) groups, notable consequences of Theorem A for the applicable G_Γ are:

Corollary. Suppose Γ is an extended Coxeter diagram with $\mathcal{S}_\Gamma^f = \mathcal{S}_\Gamma^{fs}$ which has no subdiagram of type $A_4(3)$. Then the word problem and congugacy problem for G_Γ are solvable, and G_Γ has quadratic Dehn function.

Along the way, we extract the following combinatorial criteria of Charney introduced in [Cha04] (based on [EM02]) to show that a specific 2-dimensional piecewise spherical complex is CAT(1). See Definition 5.1 and Theorem 5.2 for more detailed descriptions.

Theorem B. *Suppose Ψ is a marked A_3 simplicial complex. If Ψ is locally* CAT(1) *with the link of the $\pi/2$ vertices complete bipartite, and if the 4-cycles and 6-cycles in the 1-skeleton of Ψ seen in Figure 2 can be "filled in" to the subcomplexes in Figure 3, then Ψ is* CAT(1)*.*

Our second result is comparable to [CD95, Thm. 4.3.5]. There are certain Shephard groups where Θ_Γ has a natural cubical structure:

Theorem C. *Suppose Γ is an extended Coxeter diagram so that the underlying Coxeter diagram is type* (FC) *and $\mathcal{S}_\Gamma^f = \mathcal{S}_\Gamma^{fs}$. Then Θ_Γ is a* CAT(0) *cube complex, and hence G_Γ is (cocompactly) cubulated.*

Recall that a Coxeter diagram is type (FC) if it satisfies the following condition:

(FC) A subdiagram Γ' of Γ generates a finite Coxeter group if and only if Γ' has no edge labeled ∞.
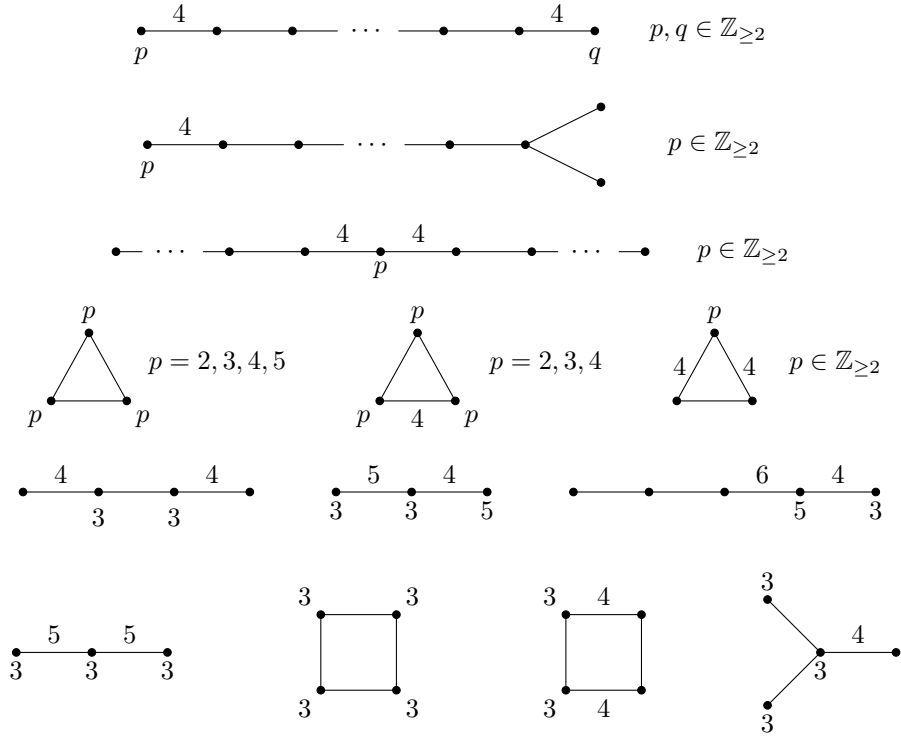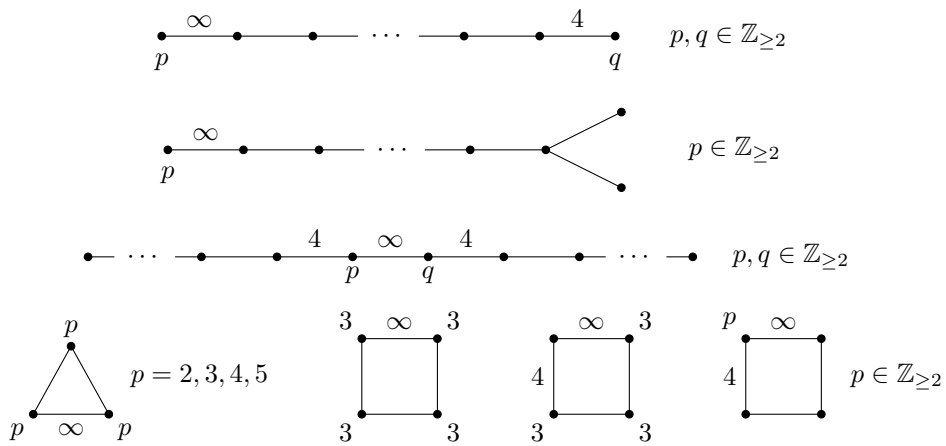
TABLE 1. Examples of CAT(0) Shephard groups

TABLE 2. Examples of cubulated Shephard groups

Thus an equivalent hypothesis for Theorem C is "Γ is an extended Coxeter diagram such that a subdiagram Γ' of Γ generates a finite Shephard group $G_{\Gamma'}$ if and only if Γ' has no edge labeled ∞."

We observe that Theorem C includes the right-angled Shephard groups (with finite vertex labels) as a special case. More examples can be found in Table 2.

The inspiration for the definition of Θ_Γ comes from [CD95], and the overarching themes are quite similar: after detailing the construction of the complexes for a given Shephard group and the metrics thereupon, we verify that links in these complexes are CAT(1). The metrics are defined identically to the Coxeter and Artin group cases—however, our technique for showing that these metrics are CAT(0) diverge greatly from [CD95], heavily utilizing the connection between the finite Shephard groups and regular complex polytopes.

1.2. **Organization of paper.** In Section 2, we establish the classification of finite Shephard groups and recall how they relate to the regular complex polytopes. In Section 3, we describe the complexes for Shephard groups which generalize the Coxeter and Deligne complexes of [CD95]. Section 4 is dedicated to proving Theorem C, which only uses the geometry and combinatorics of the regular complex polytopes. Sections 5 and 6 are dedicated to proving Theorem A; Section 5 focuses solely on proving Theorem B, while Section 6 completes the proof of Theorem A. As an aside, Appendix A compiles the various facts about the complex polytope for the "$A_3(3)$" Shephard group which are used in the proof of Theorem A.

2. Finite Shephard groups

In this section, we collect relevant background information regarding the finite Shephard groups, in particular, their classification and connection to regular complex polytopes.

2.1. **The classification of finite Shephard groups.** As discussed in the introduction, the study of Shephard groups has tended to focus on their use as finite complex reflection groups. Of course, it is well known (and one of their defining features) that the (non-Coxeter) Shephard groups arising as complex reflection groups all have a presentation as in (1.1) with diagram found in Table 4. However, if one considers an abstract group with presentation (1.1) and arbitrary diagram Γ, it is not clear if the converse holds; that is, if G_Γ is finite, then Γ appears in Tables 3 or 4. It turns out that this is in fact true, although not entirely trivial to show. To the author's knowledge, there is no published full proof of this fact (as previous work seems to only rely on the fact that the diagrams in Table 4 form a subset of the finite Shephard groups), so, we include a brief section on the proof that the diagrams in Tables 3 and 4 form the entire class of finite Shephard groups.

The following definition is adapted from [Mos80], who adapted it from the Hermitian forms typically used in the study of classical Shephard groups.

Definition 2.1. Let Γ be a Shephard group on vertex set I with all p_i finite. For an edge spanned by $i, j \in I$, $i \neq j$, define

$$\alpha_{ij} = -\left(\frac{\cos(\pi/p_i - \pi/p_j) + \cos(2\pi/m_{ij})}{2\sin(\pi/p_i)\sin(\pi/p_j)} \right)^{1/2}.$$

We let $\alpha_{ii} = 1$. If i and j are not joined by an edge, then $\alpha_{ij} = 0$.

We define a Hermitian form $H = H_\Gamma = \langle \cdot, \cdot \rangle$ on $\mathbb{C}^I$ with basis $\{ e_i : i \in I \}$ by

$$\langle e_i, e_j \rangle = \alpha_{ij}.$$

The form H viewed as a matrix (which we still call H by abuse of notation) is analogous to the cosine matrix for a usual Coxeter group, and reduces to the cosine matrix if all $p_i = 2$. We can then define a representation $\rho = \rho_\Gamma$ of G_Γ by sending the generator s_i to the complex reflection

$$R_i(z) = z + (\zeta_{p_i} - 1)\langle z, e_i \rangle e_i,$$

where $\zeta_{p_i} = \exp(2\pi\sqrt{-1}/p_i)$ is the usual primitive p_i-th root of unity. (This reduces to the complexification of the usual "canonical representation" of a Coxeter group if Γ is Coxeter.)

Our main tool is the following observation:

Proposition 2.2. *If the Shephard group G_Γ is finite, then the Hermitian form H_Γ is positive definite.*

The proof is identical to the corresponding statement for Coxeter groups. We provide the proof here for completeness. First, we need a lemma.

Lemma 2.3. (cf. [Bou02, Ch. V, §4.6 Thm. 7]) *Suppose Γ is connected. Let $H^0 = H_\Gamma^0$ denote the kernel of $H = H_\Gamma$, i.e.,*

$$H^0 = \{ x \in \mathbb{C}^I : H(x, y) = 0 \text{ for each } y \in \mathbb{C}^I \}.$$

Then G_Γ acts trivially on H^0 and every proper G_Γ-invariant subspace is contained in H^0.

Proof. By the definition of the maps R_i, if $z \in H^0$ then $R_i(z) = z$ for all i, and hence H^0 is fixed by G_Γ.

Let E be a proper subspace of $\mathbb{C}^I$ fixed by G_Γ. Suppose some e_i is contained in E. Let j be a vertex of Γ joined to i by an edge. By direct computation we see that the coefficient of e_j in $R_j(e_i)$ is nonzero, and hence e_j is also contained in E. Since Γ is connected, it follows that every basis vector is contained in E, contradicting our assumption that E is proper. Thus E contains no basis vector.

Consider the action of some R_i on $\mathbb{C}^I$. By basic linear algebra, $\mathbb{C}^I$ decomposes as the $+1$ and ζ_{p_i} eigenspaces of R_i, which are $e_i^\perp$ (the subspace orthogonal to e_i under H) and $\mathbb{C}e_i$, respectively. Since $e_i \notin E$ by assumption, we must have $E \subseteq e_i^\perp$. Thus $E \subseteq \bigcap e_i^\perp = H^0$. $\qquad \square$

We can now complete the proof of Proposition 2.2.

Proof (of Proposition 2.2). Without loss of generality, we may assume Γ is connected (otherwise H_Γ is positive definite if and only if the Hermitian form for each component is positive definite).

First we note that by classical representation theory[1], ρ is semisimple (i.e., if a subspace of $\mathbb{C}^I$ is G-invariant under ρ then it's a direct summand of $\mathbb{C}^I$) and fixes some inner product (i.e., a positive definite Hermitian form). Since this representation is semisimple, Lemma 2.3 implies H is non-degenerate; otherwise,

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[1]Finite group representations $\tau : H \to \mathbb{C}$ are semisimple by Mashke's theorem [Ser77], and fix the inner product defined by $(x, y) = \frac{1}{|H|} \sum_{h \in H} \langle \tau(h)x, \tau(h)y \rangle$, where $\langle \cdot, \cdot \rangle$ is the standard inner product on $\mathbb{C}^n$.

we would have that $H^0 \neq 0$ is a proper non-trivial subspace fixed by G, but by the Lemma, there can be no complementary G-invariant subspace of H^0, contradicting the fact that ρ is semisimple. Thus H is nondegenerate, i.e., has $H^0 = 0$, and thus by Lemma 2.3, we know that the representation ρ is irreducible. Since ρ is irreducible, it follows from Schur's lemma that H is a scalar multiple of the aforementioned invariant inner product. Since we've required $H(e_i, e_i) = 1$ for all i, it follows that H is positive definite. $\square$

Our main use for this proposition is via its contrapositive, namely that if H_Γ is not positive definite, then G_Γ is infinite. The following is the other key technical fact.

Lemma 2.4. *Let $p, q \geq 2$ and $m \geq 3$ be integers with $p = q$ whenever m is odd. Define*

$$c = -\cos(\pi/m),$$

$$\alpha = -\left(\frac{\cos(\pi/p - \pi/q) + \cos(2\pi/m)}{2\sin(\pi/p)\sin(\pi/q)} \right)^{1/2}.$$

Then $\alpha \leq c$.

Proof. We'll consider two cases. First, suppose $\alpha \leq -1$. Since cosine is bounded above by 1, this means

$$\alpha \leq -1 \leq -\cos(\pi/m) = c.$$

Now suppose $\alpha > -1$. Since $\alpha \leq 0$, this implies $\alpha^2 < 1$. By a straightforward trigonometric exercise, this happens if and only if $1/p + 1/q + 2/m > 1$. There are only finitely many such triples (p, m, q) when $m \neq 4$ (since $p = q$ when $m = 3$). These cases are easily checked by direct computation. When $m = 4$, this relation holds only when either $p = q = 3$, or $q = 2$ and $p \geq 2$ (or vice versa). If $p = q = 3$ then the computation is immediate. If $q = 2$ and $p \geq 2$, we see that

$$\begin{aligned}
\alpha &= -\left(\frac{\cos(\pi/p - \pi/2) + \cos(2\pi/4)}{2\sin(\pi/p)\sin(\pi/2)} \right)^{1/2} \\
&= -\left(\frac{\sin(\pi/p)}{2\sin(\pi/p)} \right)^{1/2} \\
&= -2^{-1/2} \\
&= -\cos(\pi/4) = c. \qquad\qquad\qquad\qquad \square
\end{aligned}$$

Proposition 2.5. *Let Γ be an extended Coxeter diagram. If the Coxeter group W_Γ is not finite, then neither is G_Γ. (In other words, $\mathcal{S}^{fs} \subseteq \mathcal{S}^f$.)*

Proof. If W_Γ is not finite, then the cosine matrix $C = C_\Gamma = (c_{ij})$, where $c_{ij} = -\cos(\pi/m_{ij})$, $c_{ii} = 1$ is not positive definite [Bou02, §4.8 Thm. 2]. This means there exists a non-zero vector $x \in \mathbb{C}^n$ such that

$$x^* C x \leq 0.$$

Expanding the product gives us

$$\sum c_{ij} \overline{x_i} x_j \leq 0.$$

By Lemma 2.4, in combination with the requirements that $\alpha_{ij} = 0$ when i and j don't span an edge of Γ and $\alpha_{ii} = 1$, we know that $\alpha_{ij} \leq c_{ij}$ for all i, j. Therefore,

$$x^* H x = \sum \alpha_{ij} \overline{x_i} x_j \leq \sum c_{ij} \overline{x_i} x_j \leq 0.$$

Hence $H = H_\Gamma$ is not positive definite, so G_Γ is not finite. $\qquad\square$

This narrows down our search to adding vertex labels to the diagrams in Table 3. The following Lemma rules out the branched Coxeter diagrams.

Lemma 2.6. *Let $D_4(p)$ denote the following extended Coxeter diagram:*
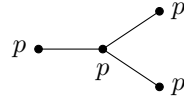
If $p > 2$, then the corresponding Hermitian form $H_{D_4(p)}$ is not positive definite. In particular, any Shephard group with a subdiagram of this form is not finite.

Proof. Let $H = H_{D_4(p)}$. Suppose the vertex set of $D_4(p)$ is $\{1, 2, 3, 4\}$, with 1 being the "central" (valence 3) vertex. We compute for $j \neq 1$ that

$$
\begin{aligned}
\alpha_{1j} = \alpha_{j1} &= -\left(\frac{\cos(\pi/p - \pi/p) + \cos(2\pi/3)}{2 \sin(\pi/p) \sin(\pi/p)} \right)^{1/2} \\
&= -\left(\frac{1 + \cos(2\pi/3)}{2 \sin^2(\pi/p)} \right)^{1/2} \\
&= -\left(\frac{1 - \frac{1}{2}}{2 \sin^2(\pi/p)} \right)^{1/2} \\
&= -\left(\frac{1}{4 \sin^2(\pi/p)} \right)^{1/2} \\
&= \frac{1}{2 \sin(\pi/p)}.
\end{aligned}
$$

By the definition of H, for all i, we have $\alpha_{ii} = 1$, and if $i \neq j$ and neither i nor j are 1, then $\alpha_{ij} = 0$. Thus the matrix representing H is

$$
\begin{pmatrix}
1 & (2\sin(\pi/p))^{-1} & (2\sin(\pi/p))^{-1} & (2\sin(\pi/p))^{-1} \\
(2\sin(\pi/p))^{-1} & 1 & 0 & 0 \\
(2\sin(\pi/p))^{-1} & 0 & 1 & 0 \\
(2\sin(\pi/p))^{-1} & 0 & 0 & 1
\end{pmatrix}.
$$

The determinant of this matrix is

$$1 - \frac{3}{4 \sin^2(\pi/p)}$$

Notice that this function is decreasing in p. When $p = 3$, this determinant is 0. Hence it is non-positive for all $p \geq 3$, and thus H is not positive definite when $p \geq 3$. It follows that any Shephard group with a $D_4(p)$ subdiagram has Hermitian form which is not positive definite, so the group is not finite. $\qquad\square$

Theorem 2.7. *A connected extended Coxeter diagram Γ produces a finite Shephard group G_Γ if and only if it appears in Tables 3 or 4.*

Proof. The result for Coxeter diagrams is well known [Bou02]. So, suppose Γ is non-Coxeter (i.e., has at least one vertex with label > 2). If Γ is a straight line, then by [Cox75, §13], G_Γ is finite if and only if it appears in Table 4. If Γ is not a straight line, then, by Proposition 2.5, it must have underlying Coxeter diagram of type D_n, $n \geq 4$, or E_m, $m = 6, 7, 8$. Since the label of vertices connected by an odd-labeled edge must agree, we have that every vertex label of Γ is p for some $p \geq 2$. Since we've assumed Γ is non-Coxeter, we know $p \geq 3$. But then Γ must contain a subdiagram of the form $D_4(p)$ with $p \geq 3$; by Lemma 2.6, this forces the group to be infinite. $\qquad\square$
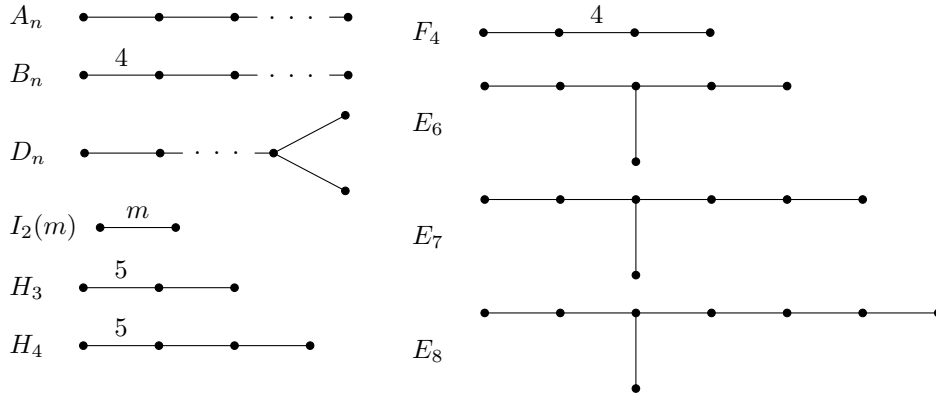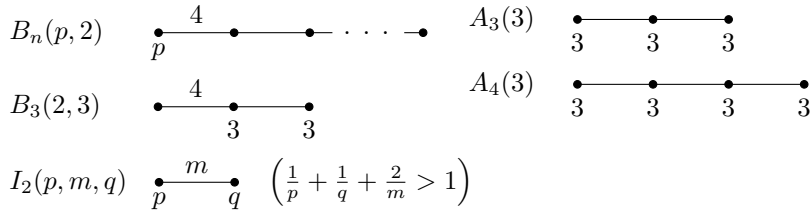

TABLE 3. The finite Coxeter diagrams

TABLE 4. The non-Coxeter finite extended Coxeter diagrams

	Shephard-Todd	Coxeter
$B_n(p,2)$	$G(p,1,n)$	$p[4]2[3]...[3]2$
$B_3(2,3)$	G_{26}	$2[4]3[3]3$
$A_3(3)$	G_{25}	$3[3]3[3]3$
$A_4(3)$	G_{32}	$3[3]3[3]3[3]3$
$I_2(p,m,q)$	various	$p[m]q$

TABLE 5. Comparison of notation

2.2. **Connection to complex polytopes.** We end this section by noting the deep relationship between the so-called regular complex polytopes and the (finite) Shephard groups, as this forms the basis for most of our arguments. First, we recall the definitions. In many ways they are identical to (reasonable) definitions of regular polytopes in $\mathbb{R}^d$, with the only substantial and necessary differences made to accomodate the fact that complex "lines" have real dimension 2. A standard reference for this material is [Cox75].

We introduce common terminology. Let $\mathcal{P}$ be any collection of affine subspaces of $\mathbb{C}^d$ (which may include the "empty subspace" $\varnothing$ and the whole space). An n-face of $\mathcal{P}$ is an n-dimensional element of $\mathcal{P}$ (by convention, $\dim(\varnothing) = -1$). Sometimes we call 0-faces the vertices of $\mathcal{P}$ and 1-faces the edges of $\mathcal{P}$. We say two faces are incident if they are distinct and one contains the other as a subspace. If $F_1 \subseteq F_2$ are incident faces and $\dim(F_1) \le \dim(F_2) - 2$, then the collection of all F' with $F_1 \subseteq F' \subseteq F_2$ is called the *medial figure* of F_1 and F_2. A symmetry of $\mathcal{P}$ is a unitary map of $\mathbb{C}^d$ which fixes the union of the elements of $\mathcal{P}$ as a set. The collection of all such symmetries is the *symmetry group* of $\mathcal{P}$. Note that a symmetry necessarily preserves incidence.

Definition 2.8. We call a collection $\mathcal{P}$ of affine subspaces of $\mathbb{C}^d$ a *regular complex (n-)polytope* if it satisfies the following axioms.

(1) $\varnothing \in \mathcal{P}$.
(2) There is unique n-face of $\mathcal{P}$ and all elements of $\mathcal{P}$ are contained in this n-face.
(3) Every medial figure contains at least two faces of each (appropriate) dimension.
(4) Every medial figure of an i-face with an j-face with $i \le j - 3$ is connected.
(5) Its symmetry group acts simply transitively on the set of maximal chains (linearly ordered sets) of faces.

A standard result is the following.

Theorem 2.9. *If $\mathcal{P}$ is a regular complex polytope, then its symmetry group is a finite Shephard group with diagram a straight line (meaning the valence of exactly two of the vertices is 1 and all other vertices have valence 2). Conversely, if Γ is a straight-line extended Coxeter diagram then G_Γ is the symmetry group of a regular complex polytope.*

This is detailed in [Cox75, §12.1]. This is not a 1-1 correspondence, as there are "starry" and "non-starry" polytopes that can be constructed for a given Shephard group. From here on, when we say "regular complex polytope" we always mean the non-starry polytope. With this restriction, the above does give a 1-1 correspondence (modulo taking dual polytopes).

We highlight specifically how a Shephard group acts on its corresponding polytope. Let Γ be a connected extended Coxeter diagram which is linear and has vertices named $1, \ldots, n$ from left to right (or right to left) with $\{s_i : i = 1, \ldots, n\}$ the generators of G_Γ. We fix an arbitrary maximal chain $F_{-1} < F_0 < F_1 < \cdots < F_n < F_{n+1}$ (where $F_{-1} = \varnothing$ and F_{n+1} is the top-dimensional face of $\mathcal{P}$) sometimes denoted by $\mathcal{F}_0$ and called the *base chain*[2]. A generator s_i acts by fixing the elements

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[2]Some authors call $\mathcal{F}_0$ the "base flag"—we use "chain" rather than "flag" as to not confuse the complex of chains with the notion of "flag complex" for an arbitrary simplicial complex.

F_j for $j \neq i$, and cyclically permuting the i-faces which are incident to each F_j, $j \neq i$. Moreover, the stabilizer of the face F_i is the subgroup of G generated by $\widehat{s_i} \coloneqq S \setminus \{s_i\}$. The following clarifies the properties of this action further.

Lemma 2.10. [ORS00, Lemma 12] *Let $F_{i_1} \subseteq F_{i_2} \subseteq \cdots \subseteq F_{i_k}$ be a subchain of $\mathcal{F}_0$, with $T = \{s_{i_1}, \ldots, s_{i_k}\}$. The stabilizer of this chain is $G_{\widehat{T}}$, where $\widehat{T} = S \setminus T$.*

We will call a G-translate of such a subchain a "chain of type $\widehat{T}$", since the stabilizer is a conguate of the subgroup of G generated by $\widehat{T}$.

The following clarifies that these stabilizers are themselves Shephard groups generated by subdiagrams of Γ.

Proposition 2.11. *Let Γ be a finite extended Coxeter diagram on vertex set I with $S = \{s_i : i \in I\}$. Let $T \subseteq S$. Then $G_T \cong G_{\Gamma(T)}$ via the natural map discussed in the introduction.*

Proof. We give a rough outline of the argument which is implicit throughout [Cox75].

If Γ is Coxeter the result is well known, so suppose it is one of the diagrams in Table 4. Let Γ' be a connected subdiagram. Name the vertices of Γ left to right as $1, \ldots, n$, and let $T = \{s_i, s_{i+1}, \ldots, s_j\}$ for some $i \leq j$. Let $\mathcal{P}_{i,j}$ denote the medial figure of F_{i-1} and F_{j+1}. A medial figure of a regular complex polytope from an $(i-1)$-face to a $(j+1)$-face is itself a regular complex polytope. Moreover, the type[3] of this medial figure is the subdiagram of Γ from i to j, which in our case is $\Gamma(T)$. This implies that the symmetry group of $\mathcal{P}_{i,j}$ is $G_{\Gamma(T)}$. But by the description of the action of G_Γ given above, we see that $\mathcal{P}_{i,j}$ is the orbit of the base chain $\mathcal{F}_0$ under G_T and, moreover, that this action is simply transitive. Thus G_T is also the symmetry group of $\mathcal{P}_{i,j}$, and since the action of the generators of G_T and $G_{\Gamma(T)}$ is identical, it can then be easily seen that $G_T \cong G_{\Gamma(T)}$.

The case where T is an arbitrary subset of S follows immediately from seeing that $G_{\Gamma(T)}$ is a direct product of groups with connected diagram, and G_T is an (internal) direct product of subsets of S of the form considered above. $\square$

Thus in the case when G_Γ is finite, we freely identify G_T and $G_{\Gamma(T)}$ and will refer to either as a (standard) parabolic subgroup.

The following standard construction was used implicitly in the previous proof, and it will be useful to have the precise statement later.

Definition 2.12. Let $\mathcal{P}_1, \mathcal{P}_2$ be regular complex polytopes in $\mathbb{C}^{d_1}$ and $\mathbb{C}^{d_2}$, respectively. Then the product polytope $\mathcal{P} = \mathcal{P}_1 \times \mathcal{P}_2$ is the collection of all subspaces $f_1 \times f_2 \subseteq \mathbb{C}^{d_1} \times \mathbb{C}^{d_2} \cong \mathbb{C}^{d_1 d_2}$ with $f_i \in \mathcal{P}_i$, ordered by inclusion. Note that $e_1 \times e_2 \subseteq f_1 \times f_2$ if and only if both $e_i \subseteq f_i$.

Repeated products are defined in the obvious way. We emphasize that for any subspace $V \subseteq \mathbb{C}^n$, we always have $\varnothing \times V = V \times \varnothing = \varnothing$. In particular, if $\mathcal{P}_{>\varnothing}$ denotes the non-empty faces of $\mathcal{P}$, then $\mathcal{P}_{>\varnothing} \times \mathcal{Q}_{>\varnothing} = (\mathcal{P} \times \mathcal{Q})_{>\varnothing}$.

3. Complexes for Shephard groups

Now we describe two complexes for a given Shephard group based on familiar definitions for Coxeter groups and Artin groups.

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[3]We've glossed over this, but it is possible to combinatorially describe a polytope via an extended Coxeter diagram without reference to symmetry group, and this is what we mean by "type".

First, we introduce common notation: if Ψ is a cell complex, we denote the closed (resp. open) star of a vertex v in Ψ by $\mathrm{St}(v) = \mathrm{St}(v, \Psi)$ (resp. $\mathrm{st}(v) = \mathrm{st}(v, \Psi)$), and the boundary of a star is $\partial\mathrm{St}(v) = \partial\mathrm{St}(v, \Psi) = \mathrm{St}(v) \setminus \mathrm{st}(v)$. The link of v in Ψ is denoted $lk(v) = lk(v, \Psi)$.

3.1. A complex for finite Shephard groups.

In this section, let Γ be an extended Coxeter diagram such that $G = G_\Gamma$ is finite. In [Orl90], an analogue of the Coxeter complex is defined for Shephard groups, there called the *Milnor fiber complex*. This complex is one of the main objects of study of this paper. For non-Coxeter finite Shephard groups, it acts as a generalization of the Coxeter complex for reasons we explain shortly. We briefly recall the definition before restating it in terms of complexes of groups. (The original definition is given in terms of invariant polynomials, but is equivalent to the following definition in terms of polytopes by [Orl90, Thm. 5.1].)

Suppose Γ is a straight line diagram (recall: Γ is connected and all vertices are valence 2 except for two unique vertices of valence 1). Let $\mathcal{P}$ be the (unique non-starry) regular complex n-polytope with G_Γ as its symmetry group. Denote by $\mathcal{P}'_p$ the derived complex of the subposet $\mathcal{P}_p$ of $\mathcal{P}$ consisting of the *proper* faces of $\mathcal{P}$; that is, the set of chains (linearly ordered subsets) of $\mathcal{P}$ excluding $\varnothing$ and $\mathbb{C}^n$ (the unique n-dimensional face of $\mathcal{P}$). This is an abstract simplicial complex, and thus has a geometric realization, which we will call the *Milnor fiber complex* or *extended Coxeter complex* $\widehat{\Theta} = \widehat{\Theta}_\Gamma = \widehat{\Theta}(\Gamma)$. Note that if Γ is a (straight line) Coxeter diagram then this is isomorphic to the usual Coxeter complex. The n-simplices of $\widehat{\Theta}$ will be denoted

$$[f_0 < f_1 < \cdots < f_n]$$

where $f_0 < f_1 < \cdots < f_n$ is a chain of faces of $\mathcal{P}$. To rephrase the definition of the derived complex, the vertices of $\widehat{\Theta}$ correspond to (proper) faces of $\mathcal{P}$, and a collection of vertices span a simplex if and only if they are linearly nested (which in turn happens if and only if they are pairwise nested, implying $\widehat{\Theta}$ is a flag complex). Thus $\widehat{\Theta}$ inherits an action of G from its action on $\mathcal{P}$. There is a "base chamber" $\mathcal{C}_0$ of $\widehat{\Theta}$ corresponding to the base chain $\mathcal{F}_0$. If $F_{i_1} < F_{i_2} < \cdots < F_{i_k}$ is a subchain of $\mathcal{F}_0$, with $T = \{s_{i_1}, \ldots, s_{i_k}\}$, then we call a G-translate of

$$[F_{i_1} < F_{i_2} < \cdots < F_{i_k}]$$

a *simplex of type* $\widehat{T}$. Since G acts simply transitively on the set of chains of $\mathcal{P}$ by assumption, $\mathcal{C}_0$ is a strict fundamental domain for the action of G on $\widehat{\Theta}$. By Lemma 2.10, the stabilizer of a simplex of type $\widehat{T}$ contained in the base chamber is $G_{\widehat{T}}$. This implies that the stabilizer of a simplex of type $\widehat{T}$ is a conjugate of $G_{\widehat{T}}$.

We can rephrase the above in the language of complexes of groups as follows.

Definition 3.1. Let Γ be any finite extended Coxeter diagram. Define $\Delta = \Delta_\Gamma = \Delta_S$ to be a simplex whose vertices are labeled by the generators S of G_Γ (equiv., the vertices of Γ). For $T \subseteq S$, let σ_T denote the face of Δ_S spanned by the elements of T. We define a complex of groups $\widehat{\mathcal{G}} = \widehat{\mathcal{G}}(G_\Gamma, \Delta_\Gamma)$ by declaring the local group at the face σ_T to be the group $G_{\widehat{T}}$. (Recall $\widehat{T} \coloneqq S \setminus T$.) The edge maps are the standard maps induced by the inclusion of generating sets.

This is a simple complex of groups, and hence its fundamental group is given by the direct limit of the system of edge maps [BH13, Def. II.12.12]. Clearly by our definitions, in this setting the fundamental group of $\widehat{\mathcal{G}}$ is G_Γ.

By our discussion above, if Γ is a straight line, this complex is developable with development $\widehat{\Theta} = \widehat{\Theta}_\Gamma$. If Γ is a Coxeter diagram, then this definition coincides with the usual definition of the Coxeter complex of W_Γ, and hence is also developable. Thus $\widehat{\mathcal{G}}$ is developable for any finite Shephard group. Sometimes when Γ is a (non-extended) Coxeter diagram, we write $\widehat{\Sigma}(\Gamma) = \widehat{\Sigma}_\Gamma = \widehat{\Theta}_\Gamma$ and call $\widehat{\Sigma}_\Gamma$ the *Coxeter complex* of Γ. It is well known that if Γ is a Coxeter diagram, then $\widehat{\Sigma}_\Gamma$ possesses a natural metric under which it is isometric to a sphere. We discuss metrics on $\widehat{\Theta}_\Gamma$ for non-Coxeter Γ in a later section.

Before concluding, we present an observation which will be of use later.

Lemma 3.2. *Let Γ be a finite extended Coxeter diagram and δ a simplex of type $\widehat{T}$. Then*

$$lk(\delta, \widehat{\Theta}) \cong \widehat{\Theta}(\Gamma(\widehat{T})) = \widehat{\Theta}(\Gamma \setminus \Gamma(T)),$$

where for a subdiagram Γ' of Γ, we let $\Gamma \setminus \Gamma'$ denote the subdiagram of Γ on vertex set $\mathrm{Vert}(\Gamma) \setminus \mathrm{Vert}(\Gamma')$.

Proof. First consider a vertex v of type $\hat{s}$. The vertex set of $lk(v, \widehat{\Theta})$ is the collection of all vertices of $\widehat{\Theta}$ which are joined to v by an edge, and incidence in the link is inherited from incidence in $\widehat{\Theta}$. These vertices are the faces of $\mathcal{P}$ which are incident to the face v. Thus there are two types of faces, those containing v and those contained by v. If a face contains v, then it's in the medial polytope between v and $\mathbb{C}^d$ which forms one component of $\widehat{\Theta}(\Gamma(\hat{s}))$. If a face is contained in v, then it's in the medial polytope between $\varnothing$ and v, which forms the other component of $\widehat{\Theta}(\Gamma(\hat{s}))$. Thus $lk(v, \widehat{\Theta}) \cong \widehat{\Theta}(\Gamma(\hat{s}))$.

Now let δ be any simplex of type $\widehat{T}$. The link of δ is

$$lk(\delta, \widehat{\Theta}) = \bigcap_{\substack{v \in \delta \\ v \text{ a vertex}}} lk(v, \widehat{\Theta}),$$

with intersection interpreted on the level of posets. A vertex v' of $\widehat{\Theta}$ is in each $lk(v, \widehat{\Theta})$ if and only if it is a face in each $\widehat{\Theta}(\Gamma(\hat{s}))$, $s \in T$, under the identification made in the previous paragraph. It is easy to see that this holds if and only if

$$v' \in \widehat{\Theta}\left(\bigcap_{s \in T} \Gamma(\hat{s})\right)$$

$$= \widehat{\Theta}\left(\Gamma\left(\bigcap_{s \in T} \hat{s}\right)\right)$$

$$= \widehat{\Theta}(\Gamma(\widehat{T})). \qquad \square$$

3.2. A complex for arbitrary Shephard groups. Throughout this section, let Γ be any extended Coxeter diagram on vertex set I with $S = \{s_i : i \in I\}$ the standard generators for $W = W_\Gamma$ and $G = G_\Gamma$.

The following definition is based on the definition of the "modified Coxeter/Deligne complexes" of [CD95].

Definition 3.3. Let $K = K_\Gamma = |(\mathcal{S}_\Gamma^{fs})'|$, where $(\mathcal{S}_\Gamma^{fs})'$ denotes the derived complex of $\mathcal{S}_\Gamma^{fs}$ and $|(\mathcal{S}_\Gamma^{fs})'|$ is its geometric realization. An n-simplex of K is denoted

$$[T_0 < T_1 < \cdots < T_n]$$

for a chain $T_0 < T_1 < \cdots < T_n$ with each $T_i \in \mathcal{S}^{fs}$. In particular, the vertices are indexed by elements $T \in \mathcal{S}^{fs}$; we let $v_T = [T]$ denote the vertex of K coming from T. We define a complex of groups $\mathcal{G} = \mathcal{G}(G_\Gamma, K_\Gamma)$ over K by declaring the local group at v_T to be $G_{\Gamma(T)}$ and the edge maps to be the natural maps coming from the inclusion of generators.

As with the complex for the finite Shephard groups, this is also a simple complex of groups. Hence by [BH13, Def. II.12.12], the fundamental group of $\mathcal{G}$ is the direct limit over the edge maps, and when $\mathcal{S}^f = \mathcal{S}^{fs}$, this is clearly G_Γ.

If Γ is a Coxeter diagram, this complex of groups is developable [CD95], with development typically denoted $\Sigma = \Sigma_\Gamma$, called the *modified Coxeter complex* or the *Davis complex*. More generally, if $\mathcal{G}$ is developable, we will denote its development $\Theta = \Theta(\Gamma) = \Theta_\Gamma$ and may appropriately refer to it as the *modified extended Coxeter complex*, *modified Milnor fiber complex*, or *extended Davis complex*.

A priori, it is not known in general if $\mathcal{G}$ is developable for an arbitrary extended Coxeter diagram. The rest of the paper is dedicated to studying developability and, more specifically, non-negative curvature of $\mathcal{G}$, utilizing the following (paraphrased) lemma.

Lemma 3.4. [BH13, Thm. II.12.28] *If $\mathcal{H}$ is a (simple) complex of groups over a simply connected domain and the local development at each vertex is locally* CAT(0), *then $\mathcal{H}$ is developable and has locally* CAT(0) *development.*

It is clear that K is simply connected ($[\varnothing]$ is a cone point). So in order to make use of this, we need a metric on K, which we will discuss shortly. First, we describe a cell structure on K which is coarser than the given simplicial structure. The following definitions again are based on [CD95].

Definition 3.5. For $T \in \mathcal{S}^{fs}$, let

$$\mathcal{S}_{\geq T}^{fs} = \{\, T' \in \mathcal{S}^{fs} : T' \supseteq T \,\} \qquad \mathcal{S}_{>T}^{fs} = \{\, T' \in \mathcal{S}^{fs} : T' \supsetneq T \,\}$$
$$\mathcal{S}_{\leq T}^{fs} = \{\, T' \in \mathcal{S}^{fs} : T' \subseteq T \,\} \qquad \mathcal{S}_{<T}^{fs} = \{\, T' \in \mathcal{S}^{fs} : T' \subsetneq T \,\}$$
$$F_T = |(\mathcal{S}_{\geq T}^{fs})'| \qquad\qquad F_T^\circ = |(\mathcal{S}_{\geq T}^{fs})'| \setminus |(\mathcal{S}_{>T}^{fs})'|$$
$$F_T^* = |(\mathcal{S}_{\leq T}^{fs})'| \qquad\qquad (F_T^*)^\circ = |(\mathcal{S}_{\leq T}^{fs})'| \setminus |(\mathcal{S}_{<T}^{fs})'|.$$

We sometimes call F_T (F_T°) a *face* (resp. *open face*) of K and F_T^* (($F_T^*)^\circ$) a *dual face* (resp. *open dual face*) of K.

Notice that F_T^* is combinatorially a cube—the faces of the cube are of the form $F_{T_1} \cap F_{T_2}^*$ where $T_1 \subseteq T_2 \subseteq T$. So K itself has a cubical structure where the cubical faces are $F_{T_1} \cap F_{T_2}^*$ for $T_1 \subseteq T_2 \in \mathcal{S}^{fs}$. The following is a standard exercise.

Proposition 3.6. *With this cell structure, F_T^* is isomorphic to the cone on the barycentric subdivision Δ_T' of Δ_T with cone point v_T. The isomorphism is given by sending a vertex $v_{T'}$ to the barycenter of $\sigma_{T \setminus T'}$ in Δ_T.*

From here on, we identify $lk(v_T, F_T^*)$ and Δ_T in this way. With this connection between K and Δ_T for $T \in \mathcal{S}^{fs}$, we define the following metrics.

Definition 3.7.   (1) (The cubical metric) Since the faces of K are combinatorial
        cubes, metrize them to be standard Euclidean unit cubes. Under this metric,
        Δ_T is a spherical simplex with edge lengths all equal to $\pi/2$.
   (2) (The Moussong metric) The definition of this metric is more involved and
        we refer to [CD95, §4.4] for details. In summary, a cell $F_{T_1} \cap F^*_{T_2}$ is metrized
        to be a *Coxeter block*, which is (the closure of) a connected component of
        the Coxeter zonotope associated to the finite Coxeter group W_{T_2} minus its
        reflection hyperplanes. In this metric, if $T \in \mathcal{S}^{fs}$, then Δ_T is a spherical
        simplex where the length of the edge between the vertices corresponding to
        s_i, s_j is $\pi - \pi_{m_{ij}}$.

We conclude by briefly recalling the definition of the local development as it
applies here (our notation will differ slightly from [BH13]). In our case, the local
development of $\mathcal{G}$ further relates $\mathcal{G}$ with $\widehat{\mathcal{G}}$.

Let v_T be a vertex of K, with $T \in \mathcal{S}^{fs}$. The *upper star* St^T of v_T in $\mathcal{G}$ is the
(full) subcomplex of K spanned by the vertices $v_{T'}$ with $T' \supseteq T$. The *lower link*
Lk_T of v_T in $\mathcal{G}$ is the development of the subcomplex of groups $\widehat{\mathcal{G}}(K_{<T})$ of $\mathcal{G}(K)$,
where $K_{<T}$ denotes the subcomplex spanned by vertices $v_{T'}$ with $T' \subsetneq T$. Both of
these objects are simplicial complexes which inherit the metric placed on K. Then
the *local development at* v_T is (combinatorially) the join

$$D(T) = St^T * Lk_T.$$

Its metric naturally comes from the metric on K. The link of v_T in the local
development is

$$lk(v_T, D(T)) = Lk^T * Lk_T,$$

where Lk^T is the *upper link*, meaning the (full) subcomplex of K spanned by the
vertices $v_{T'}$ with $T' \supseteq T$. We may also sometimes refer to this complex as $K_{>T}$.

Note that $K_{>T}$ is isomorphic to $lk(v_T, F_T)$ and $K_{<T}$ is isomorphic to $lk(v_T, F^*_T)$.

We use the previous proposition to identify $K_{<T}$ with Δ_T. With this identification,
the complex of groups $\widehat{\mathcal{G}}(K_{<T})$ is isomorphic to $\widehat{\mathcal{G}}(G_T, \Delta_T)$ as defined before and
thus Lk_T is isomorphic to $\widehat{\Theta}_{\Gamma(T)}$. It is straightforward to check that the metrics
placed on K above agree with the claimed metrics on Δ_T. This is summarized in
the following

Proposition 3.8. *If Γ is any extended Coxeter diagram, then in either metric,
the link of a vertex v_T in the local development of $\mathcal{G}(G_\Gamma, K_\Gamma)$ is isometric to the
spherical join*

$$lk(v_T, F_T) * \widehat{\Theta}_{\Gamma(T)}.$$

Showing the local development is nonpositively curved amounts to showing
that these links are CAT(1). Since a spherical join is CAT(1) if and only if both
components are [BH13, Cor. II.3.15], this reduces to showing that $lk(v_T, F_T)$ and
$\widehat{\Theta}_{\Gamma(T)}$ are CAT(1) when $T \in \mathcal{S}^{fs}$. We also have the following useful corollary for
dealing with disconnected diagrams.

Corollary 3.9. *If Γ_1, Γ_2 are extended Coxeter diagrams and $\Gamma = \Gamma_1 \sqcup \Gamma_2$ is their
disjoint union, then $\mathcal{G}(G_\Gamma) \cong \mathcal{G}(G_{\Gamma_1}) \times \mathcal{G}(G_{\Gamma_2})$. In particular,*
   (1) *If $\mathcal{G}(G_\Gamma)$ is developable, then under either metric, $\Theta(\Gamma)$ is isometric to
        $\Theta(\Gamma_1) \times \Theta(\Gamma_2)$, and*

(2) *If G_Γ is finite, then under either metric, $\widehat{\Theta}(\Gamma)$ is isometric to the spherical join $\widehat{\Theta}(\Gamma_1) * \widehat{\Theta}(\Gamma_2)$.*

Proof. By a product $\mathcal{G}(G_1, K_1) \times \mathcal{G}(G_2, K_2)$ of (simple) complexes of groups, we mean the complex of groups over $K_1 \times K_2$ with vertex groups $G_{v_1}^{(1)} \times G_{v_2}^{(2)}$ for $v_i \in \text{Vert}(K_i)$ and vertex group $G_{v_i}^{(i)}$ of $\mathcal{G}(G_i, K_i)$, and edge maps $\psi_{jk}^{(1)} \times \psi_{\ell n}^{(2)}$ with $\psi_{jk}^{(i)} : G_k^{(i)} \to G_j^{(i)}$ the edge maps of $\mathcal{G}(G_i, K_i)$.

It is clear from the definitions that if $\Gamma = \Gamma_1 \sqcup \Gamma_2$ then $G_\Gamma \cong G_{\Gamma_1} \times G_{\Gamma_2}$ and $K_\Gamma \cong K_{\Gamma_1} \times K_{\Gamma_2}$. Thus we easily see that $\mathcal{G}(G_\Gamma) \cong \mathcal{G}(G_{\Gamma_1}) \times \mathcal{G}(G_{\Gamma_2})$. The statement about developability follows readily from this decomposition of the complexes of groups.

Now suppose G_Γ is finite and $\Gamma = \Gamma_1 \sqcup \Gamma_2$. In particular, $\mathcal{G}(G_\Gamma)$ is developable since G_Γ is finite. Let S be the standard generating set of G_Γ. By our definitions, $K_{>S}$ is empty; hence $\widehat{\Theta}_\Gamma \cong K_{>S} * \widehat{\Theta}_\Gamma$. By Proposition 3.8, this is link of v_S in the local development of $\mathcal{G}(G_\Gamma)$. But since the complex of groups is developable, this is simply $lk(v_S, \Theta_\Gamma)$. Since $\Theta(\Gamma) \cong \Theta(\Gamma_1) \times \Theta(\Gamma_2)$ by our previous result,

$$
\begin{aligned}
\widehat{\Theta}_\Gamma &\cong lk(v_S, \Theta_\Gamma) \\
&\cong lk(v_S, \Theta_{\Gamma_1} \times \Theta_{\Gamma_2}) \\
&\cong lk(v_S, \Theta_{\Gamma_1}) * lk(v_S, \Theta_{\Gamma_2}) \\
&\cong \widehat{\Theta}_{\Gamma_1} * \widehat{\Theta}_{\Gamma_2}. \qquad \qquad \square
\end{aligned}
$$

4. THE CUBICAL METRIC

In this section, we complete the proof of Theorem C. Assume that $\mathcal{S}^{fs} = \mathcal{S}^f$ and the underlying Coxeter diagram of Γ satisfies (FC). Endow $\mathcal{G} = \mathcal{G}(G_\Gamma, K_\Gamma)$ with the cubical metric. As discussed at the end of the previous section, we must show that $lk(v_T, F_T)$ and $\widehat{\Theta} = \widehat{\Theta}_{\Gamma(T)}$ are CAT(1) when $T \in \mathcal{S}^{fs}$.

The cubical metric places a length of $\pi/2$ on each edge of $\widehat{\Theta}$. We apply Gromov's link condition, which says that a piecewise spherical simplicial complex with edge lengths all $\pi/2$ is CAT(1) if and only if it's a flag complex (see [BH13, Thm. II.5.18] or [Gro87]).

Suppose $v_1 = [f_1], \ldots, v_n = [f_n]$ are vertices of $\widehat{\Theta}$ which are pairwise connected by edges. This means each f_i is a face of the polytope corresponding to $G_{\Gamma(T)}$, and the f_i are pairwise nested. The only way a collection of affine subspaces of $\mathbb{C}^n$ can be pairwise nested is if they are linearly nested, resulting in a simplex $[f_1 < \cdots < f_n]$ (up to change of indices) spanned by the vertices v_i. Hence $\widehat{\Theta}$ is flag, and therefore CAT(1) under the cubical metric.

We now turn to $lk(v_T, F_T)$. As with $\widehat{\Theta}$, the cubical metric places a length of $\pi/2$ on each edge, so we must show that this is a flag complex. We can view $\mathcal{S}_{>\varnothing}^{fs}$ as an abstract simplicial complex with vertex set S, with a set T spanning a simplex if and only if $W_{\Gamma(T)}$ is finite. Since the underlying Coxeter diagram of Γ satisfies (FC), this is a flag complex. By the same argument in [CD95, Lem. 4.3.3], we have that $lk(v_T, F_T) \cong lk(T, \mathcal{S}_{>\varnothing}^{fs})$. Since the link of a flag complex is still flag, and our previous remarks show that $\mathcal{S}_{>\varnothing}^{fs}$ is flag, it follows that $lk(v_T, F_T)$ is a flag complex, and hence CAT(1).

Therefore $\mathcal{G}$ is nonpositively curved, and hence developable with development $\Theta = \Theta_\Gamma$. As discussed before, Θ has the cell structure and metric of a cube complex.

Since $\mathcal{G}$ is a simple complex of groups over a simply connected domain, we know that Θ is simply conneted and hence is a CAT(0) cube complex. Notice that G_Γ acts cocompactly on Θ with fundamental domain K. The stabilizer of a cell of K is the finite group $G_{\Gamma(T)}$ for some $T \in \mathcal{S}^{fs}$, implying the stabilizer of an arbitrary cell is a conjugate of this group, and hence finite. Thus G also acts properly on Θ, and therefore is cocompactly cubulated.

5. A CAT(1) CRITERION

Before proving Theorem A, we present a criterion for a simplicial complex made of A_3-simplices to be CAT(1). The idea behind this criteria comes from [Cha04] (which itself is based on [EM02]), where it is used implicitly to show that the Moussong metric for the Deligne complex for the 4-strand braid group is CAT(0).

Definition 5.1. Let Δ be a spherical 2-simplex with vertices labeled $\hat{a}$, $\hat{b}$, $\hat{c}$ so that the angle at $\hat{a}$ and $\hat{c}$ is $\pi/3$ and the angle at $\hat{b}$ is $\pi/2$. For $g = a, b, c$, we label the edge opposite $\hat{g}$ as g. We call Δ the *marked A_3 simplex*.

We call a homogeneous[4] 2-dimensional simplicial complex Ψ a *marked A_3 simplicial complex* if every top dimensional simplex σ is endowed with a choice of isomorphism $m_\sigma : \sigma \to \Delta$ so that if $x \in \sigma \cap \sigma'$, then $m_\sigma(x) = m_{\sigma'}(x)$. The maps m_σ are the *markings* of Ψ. Sometimes we will call a top dimensional simplex a *chamber*. We call a vertex v of Ψ type $\hat{a}$, $\hat{b}$, or $\hat{c}$ if for some (hence any) chamber σ containing v, the vertex $m_\sigma(v)$ of Δ is labeled $\hat{a}$, $\hat{b}$, $\hat{c}$, respectively. Similarly, we call an edge e type a, b, c if $m_\sigma(e)$ is the edge of Δ labeled a, b, c, resp. Every marked A_3 simplicial complex has a metric, which we call the *canonical metric*, obtained by pulling back the metric of Δ along the markings, so that each chamber is a simplex of shape A_3.

If Ψ is a marked A_3 simplicial complex, we say it satisfies Charney's combinatorial CAT(1) criteria (CCCC) if

(1) The link of a vertex of type $\hat{a}$ or $\hat{c}$ has girth[5] at least 6,
(2) The link of a vertex of type $\hat{b}$ is a complete bipartite graph which contains an embedded 4-cycle,
(3) Any edge path in Figure 2 is contained in a corresponding subcomplex of Ψ shown in Figure 3. (A shaded triangle represents a 2-simplex.)

Our main theorem for this section is the following.

Theorem 5.2. *If Ψ is a marked A_3 simplicial complex which satisfies CCCC, then Ψ is* CAT(1) *under its canonical metric.*

Before proving this, we need to establish some facts about the geometry of Ψ under its canonical metric which follow from the above criteria.

One of the main consequences (and inspirations) of our definition of a marked A_3 simplicial complex is that we may *develop* certain geodesics from Ψ onto the A_3 Coxeter complex $\widehat{\Sigma} = \widehat{\Sigma}(A_3)$. Let $W = W_{A_3}$ be the Coxeter group with diagram A_3 and generators a, b, c, i.e.,

$$W = \langle\, a, b, c \mid a^2 = b^2 = c^2 = (ab)^3 = (bc)^3 = (ca)^2 = 1 \,\rangle.$$

---

[4]Recall that an n-dimensional homogeneous simplicial complex is one where every simplex is contained in some n-simplex.

[5]The complex Ψ is 2-dimensional, so the links of vertices are (simplicial) graphs. We often use the language of graph theory when referring to these links.
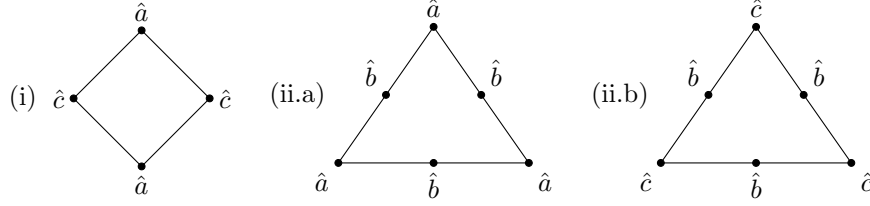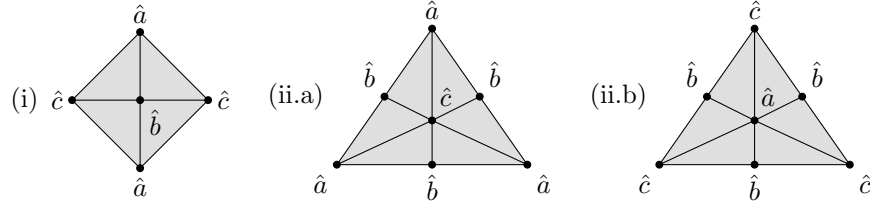
FIGURE 2. The short closed edge paths

FIGURE 3. Filling the short closed edge paths

The vertices of $\widehat{\Sigma}$ are cosets of the form $wW_{\{a,b,c\}\setminus\{g\}}$, where $g \in \{a,b,c\}$. If we define $\hat{g} := \{a,b,c\} \setminus \{g\}$, then the vertices are cosets of $W_{\hat{g}}$ for $g \in \{a,b,c\}$ and may sensibly be called *type $\hat{g}$* (i.e., type $\hat{a}$, $\hat{b}$, or $\hat{c}$). Each chamber of $\widehat{\Sigma}$ is a simplex of type A_3, such that the angle of every simplex at a vertex of type $\hat{a}$ or $\hat{c}$ is $\pi/3$, and the angle of every simplex at a vertex of type $\hat{b}$ is $\pi/2$—hence the definition of the marked A_3 simplex Δ and its metric.

Suppose γ is a local geodesic of Ψ which does not intersect any vertices. Then the sequence of chambers which intersect γ can be mapped down to a sequence of adjacent chambers of $\widehat{\Sigma}$ so that the markings of $\widehat{\Sigma}$ agree with the markings of Ψ. In particular, the image of γ is a local geodesic in $\widehat{\Sigma}$, called the *development of γ (onto $\widehat{\Sigma}$)*.

Condition (2) allows us to develop a local geodesic which intersects vertices of type $\hat{b}$ in the following way. First we note that since Ψ is marked, (2) implies that the link of a vertex of $\hat{b}$ is the join of a set of at least two vertices of type $\hat{a}$ with a set of at least two vertices of type $\hat{c}$. If γ is a local geodesic which passes through a vertex v of type $\hat{b}$, then it intersects the ε-sphere of v at two points which are distance at least π apart in the spherical metric on the ε-sphere. This sphere is isometric to the link of v, so the points on the sphere correspond to points in the link which are distance no less than π apart. Since the link is complete bipartite, they are exactly distance π apart and are contained in a loop of edge length 4 (since such a loop exists). This loop corresponds to 4 simplices of Ψ which form a quadrilateral as in Figure 4. These simplices can then be mapped down to $\widehat{\Sigma}$, and the image of γ will still be locally geodesic at the image of v.

In fact, any local geodesic which passes through the (open) star of a vertex of type $\hat{b}$ is contained in a quadrilateral of this form. (If it doesn't pass through the
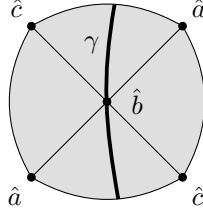
FIGURE 4. Developing a geodesic through a vertex of type $\hat{b}$

vertex, this follows from the usual developing process, plus condition (2).) This leads us to the following

Lemma 5.3. *Suppose v is a vertex of type $\hat{b}$ and $\gamma : [a_1, a_2] \to \mathrm{St}(v)$ is a geodesic segment with endpoints $v_i = \gamma(a_i)$ on the interior of an edge e_i of $\partial\mathrm{St}(v)$. Then either e_1 and e_2 are adjacent (in which case we say γ "cuts a corner" of $\mathrm{St}(v)$), or there is another edge e_0 adjacent to both e_1 and e_2 (in which case we say γ "traverses" $\mathrm{St}(v)$).*
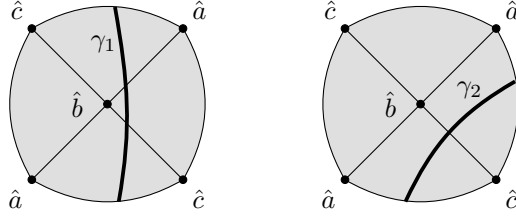
FIGURE 5. Traversing (γ_1) and cutting a corner (γ_2) of a $\hat{b}$ star

This follows easily from examining the aforementioned quadrilateral in the Coxeter complex. See Figure 5. The following can be seen in a similar way by developing to the Coxeter complex.

Lemma 5.4. *Let γ be a local geodesic of Ψ avoiding vertices of type $\hat{a}$ and $\hat{c}$. If γ traverses (resp, cuts a corner of) some star of a vertex of type $\hat{b}$, then it traverses (resp, cuts a corner of) every star of a vertex of type $\hat{b}$ whose interior intersects γ.*

The following is one of the main lemmas of this section. It allows us to place a lower bound on the length of geodesics which avoid $\hat{a}$ and $\hat{c}$ vertices, as well as allows us to fill in other short loops.

Lemma 5.5. *The (closed) stars of two distinct vertices of type $\hat{b}$ intersect trivially, in exactly one vertex, or in exactly one edge.*

Proof. Let v_1, v_2 be distinct vertices of type $\hat{b}$. We first show that $\mathrm{St}(v_1) \cap \mathrm{St}(v_2)$ contains at most one vertex of type $\hat{a}$ and at most one vertex of type $\hat{c}$.

Let $w_1, w_2 \in \mathrm{St}(v_1) \cap \mathrm{St}(v_2)$ be vertices of type $\hat{a}$. This results in the leftmost edge path in Figure 6. By condition (2), there are vertices $u_i \in \mathrm{St}(v_i)$ of type $\hat{c}$
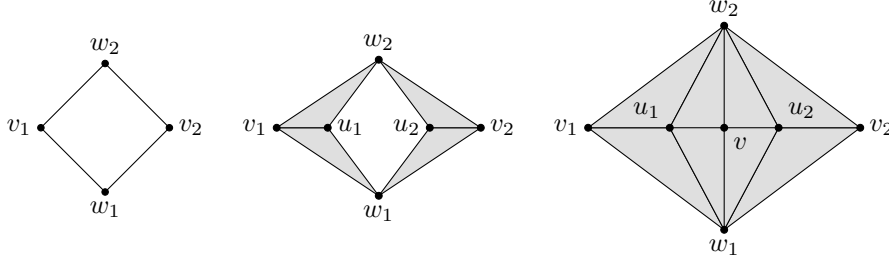
FIGURE 6. Filling a loop coming from the intersection of two stars

so that u_i is adjacent to w_1, w_2 for $i = 1, 2$. This results in the middle complex of
Figure 6. The loop consisting of the w_i and u_i is a loop of type (i) (in Figure 2)
and thus there is a vertex v of type $\hat{b}$ so that the middle complex of Figure 6 can be
augmented to the rightmost complex.

If $v \neq v_1$ and $w_1 \neq w_2$, then the vertices v_1, w_1, v, w_2 give rise to an embedded
loop of length 4 in the link of u_1. However, condition (1) prevents this from
happening, so we must have either $v = v_1$ or $w_1 = w_2$. The same reasoning shows
that either $v = v_2$ or $w_1 = w_2$. If $w_1 \neq w_2$, we would have $v_1 = v = v_2$, contradicting
the assumption that $v_1 \neq v_2$. So, we must have $w_1 = w_2$ and thus $\mathrm{St}(v_1) \cap \mathrm{St}(v_2)$
contains at most one vertex of type $\hat{a}$. An analagous argument shows that it contains
at most one vertex of type $\hat{c}$.

Now suppose $w, u \in \mathrm{St}(v_1) \cap \mathrm{St}(v_2)$ are vertices of type $\hat{a}$ and $\hat{c}$, respectively.
As noted before, the definition of marked A_3 simplicial complex and condition (2)
imply that the link of a vertex of type $\hat{b}$ is a join of a set of vertices of type $\hat{a}$ with a
set of vertices of type $\hat{c}$. This means there is an edge, which is necessarily unique,
between w and u, and this edge contained in $\mathrm{St}(v_1) \cap \mathrm{St}(v_2)$.

The result follows from noticing that if e_1, e_2 are two distinct edges, then their
union contains at least three vertices (since Ψ is simplicial), and thus $\mathrm{St}(v_1) \cap \mathrm{St}(v_2)$
cannot contain each of these vertices by our above work, meaning at least one of
the edges is not contained in this intersection.   □

As a first consequence, we have

Lemma 5.6. *Any edge path appearing in Figure 7 is contained in a subcomplex of
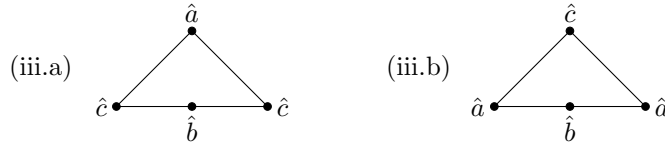Ψ found in Figure 8.*

FIGURE 7. Two more short closed edge paths

Proof. We will deal with the left loop of Figure 7; the argument for the other loop
is identical. By condition (2), we know that this loop is a subcomplex (indicated in
bold) of the leftmost complex in Figure 9, since two $\hat{c}$ vertices in a star of a $\hat{b}$ vertex
must be connected by an $\hat{a}$ vertex (which is also in this star). But now the vertices
of type $\hat{a}$ and $\hat{c}$ form a loop of type (i) (Figure 2), and in particular are contained

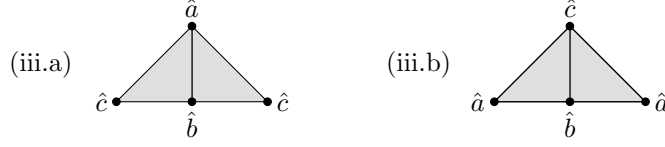(iii.a)   (iii.b)

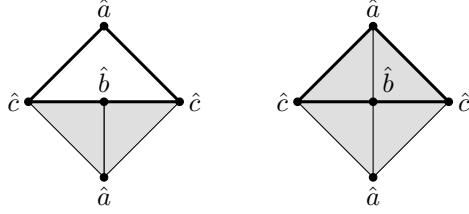FIGURE 8. Filling the edge paths

FIGURE 9. Completing the loop

in the star of a common $\hat{b}$ vertex. If this vertex were distinct from the one already included in our original loop, this would contradict Lemma 5.5. So the complex on the left of Figure 9 can be completed to the complex on the right, giving the desired result. □

We now show that the loops considered so far are the only possible short loops that can arise.

Lemma 5.7. *Suppose p is a simple closed edge path of Ψ. Then $\ell(p) < 2\pi$ (where ℓ is the length function) if and only if p is one of the paths of type (i), (ii), or (iii) in Figures 2 or 7.*

Proof. An edge of p from a vertex of type $\hat{a}$ (or $\hat{c}$) to $\hat{b}$ is immediately followed by an edge from a vertex of type $\hat{b}$ to $\hat{a}$ (resp. $\hat{c}$). This is a consequence of condition (2) and the fact that the length of edges of the link is $\pi/2$. Moreover, as a consequence of the definition of a marked A_3 simplicial complex, vertices of type $\hat{a}$ or $\hat{c}$ cannot be joined by an edge to a vertex of the same type. Thus p consists of edges between $\hat{a}$ vertices and $\hat{c}$ vertices, the length of which is $\beta := \arccos(1/\sqrt{3})$, and pairs of edges from $\hat{a}$ to $\hat{b}$ to $\hat{a}$, or from $\hat{c}$ to $\hat{b}$ to $\hat{c}$, the total length of which is 2α, where $\alpha := \arccos(1/3)$. We conclude by an argument completely identical to that found in [Cha04, Pf. of Thm. 5.4] that the only possible edge paths of length less than 2π are those in Figures 2 or 7 (which are those of Figure 7 of [Cha04]), as desired. □

We are now ready to prove the main theorem of the section. The argument is completely analogous to that of [Cha04, Pf. of Thm. 5.4]. We give the argument here to clarify that it works in this more general setting.

Proof (of Theorem 5.2). First we note that the links of vertices are CAT(1). For a vertex v of type $\hat{a}$ or $\hat{c}$, the canonical metric places a length of $\pi/3$ on the edges of $lk(v, \Psi)$. Condition (1) guarantees the (combinatorial) girth of this link is 6, and hence the length of any nontrivial closed loop in $lk(v, \Psi)$ is at least $6\pi/3 = 2\pi$. If v

is a vertex of type $\hat{b}$, then the canonical metric places a length of $\pi/2$ on the edges of $lk(v, \Psi)$. Since this link is bipartite, it has girth at least 4, so the length of any nontrivial closed loop in $lk(v, \Psi)$ is at least $4\pi/2 = 2\pi$. Hence the vertex links are CAT(1). It remains to show that if γ is a closed geodesic of Ψ, then $\ell(\gamma) \geq 2\pi$. So, suppose γ is a closed geodesic. There are three cases to consider.

First assume γ is an edge path. If $\ell(\gamma) < 2\pi$, then Lemma 5.7 implies γ is one of the paths of type (i), (ii), or (iii) (Figures 2 and 7). Condition (3) (resp. Lemma 5.6) guarantees that the paths of type (i) and (ii) (resp. (iii)) are not locally geodesic at any vertex of type $\hat{a}$ or $\hat{c}$, and thus we must have $\ell(\gamma) \geq 2\pi$.

Now assume γ intersects the interior of at least one 2-cell and does not intersect any vertices of type $\hat{a}$ or $\hat{c}$. Let $v_1, \ldots, v_n$ denote the distinct vertices of type $\hat{b}$ such that $\gamma \cap st(v_i) \neq \varnothing$. Since such a star is CAT(1) and has diameter $< \pi$, we know $n \neq 1$. By Lemmas 5.3 and 5.5, γ cannot close up after intersecting only two such stars, so $n \neq 2$. By Lemma 5.4, there are two cases now to consider; either γ traverses every $\mathrm{St}(v_i)$ or cuts a corner of every $\mathrm{St}(v_i)$.
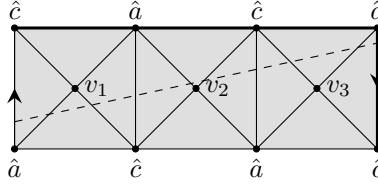

FIGURE 10. The case $n = 3$

Suppose first that γ traverses these stars. We claim that $n \neq 3$. To the contrary, if $n = 3$ then we have a subcomplex of Ψ seen in Figure 10, with γ indicated by a dashed line and the edges marked with arrows identified. This results in an edge path in the union of the $\mathrm{St}(v_i)$ which forms a path of type (i) above but is not contained in any of the $\mathrm{St}(v_i)$, indicated in bold in Figure 10. By condition (3), it follows that this path is contained in some $\mathrm{St}(v')$ for a vertex v' of type $\hat{b}$ distinct from each v_i. But then $\mathrm{St}(v')$ would intersect $\mathrm{St}(v_3)$ in more than one edge, contradicting Lemma 5.5. By developing onto $\widehat{\Sigma}$, any geodesic which traverses 4 quadrilaterals of the type in Figure 4 has length no less than 2π.

Now suppose γ cuts corners. We claim $n \geq 6$, resulting in a complex of the form in Figure 11, with γ represented by a dashed line. If γ closes up after cutting less than 6 corners, then two of the three circled vertices in Figure 11 would be identified, which cannot happen (either because Ψ is simplicial and each simplex in the diagram is distinct, or because of Lemma 5.5). Hence we must have $n \geq 6$. By developing, a geodesic which cuts at least 6 corners has length $\geq 2\pi$.

Last, assume γ intersects at least one vertex of type $\hat{a}$ or $\hat{c}$ but is not an edge path. Decompose γ into the concatenation of segments $\gamma = \gamma_0 \gamma_1 \ldots \gamma_n$ with no vertices of type $\hat{a}$ or $\hat{c}$ in the interior of the γ_i. Note that at least one of the γ_i contains no edges of Ψ. By developing we see that any such γ_i is half a great circle from a vertex of type $\hat{a}$ or $\hat{c}$ to a vertex of type $\hat{c}$ or $\hat{a}$, respectively, so the length of γ_i is π—see Figure 12. Thus if there are two segments of this type, $\ell(\gamma) \geq 2\pi$, so suppose there is exactly one segment γ_i which does not contain an edge of Ψ. We may assume γ_0 is this segment. Notice in particular that since the endpoints of γ_0 always have different type, γ_0 cannot be closed, or in other words $n > 0$.
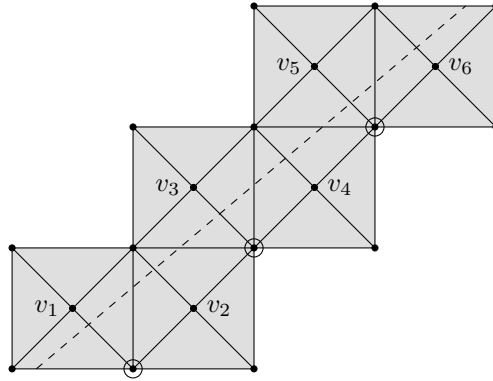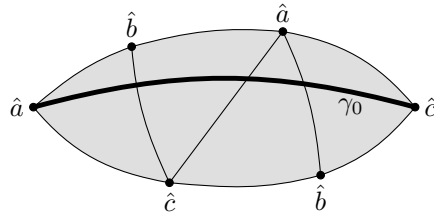
FIGURE 11. A geodesic cutting corners

FIGURE 12. A geodesic segment γ_0 in Ψ which is not an edge path

Let $\overline{\gamma}_0$ be the image of γ_0 under the development to $\widehat{\Sigma}$. Rotate $\overline{\gamma}_0$ within $\widehat{\Sigma}$ relative to its endpoints until it becomes an edge path with vertices of type $\hat{a}$ or $\hat{c}$ in its interior. Then lift this rotation to γ_0' in Ψ so that its endpoints agree with the original endpoints of γ_0 (see Figure 13). This new path is clearly locally geodesic in its interior and has the same length as γ_0. Let $\gamma' = \gamma_0'\gamma_1 \ldots \gamma_n$. Note that $\ell(\gamma') = \ell(\gamma)$. We claim that γ' is locally geodesic. To do this, we only need to check the endpoints of γ_0'.
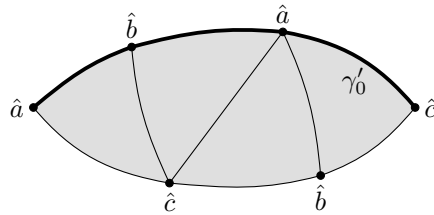
FIGURE 13. A possible rotation of γ_0 in Ψ

Since γ_0 is the unique non-edge path in the decomposition of γ, we know that γ_1 and γ_n are both edge paths in Ψ. (The case $n = 1$ is acceptable.) Let v_0 be the vertex between γ_0 and γ_1, and let v_n be the vertex between γ_n and γ_0 (if $n = 1$, let these vertices be the distinct endpoints of γ_0). Consider $lk(v_j, \Psi)$ for $j = 1$ and $j = n$. The intersection of the ε-sphere of v_j with γ_0 and γ_j gives points in $lk(v_j, \Psi)$

which are distance at least π apart. Since γ_j is an edge path, it gives a vertex v'_j of $lk(v_j, \Psi)$, and since γ_i is not an edge path it gives a point p_j in the interior of an edge e_j of $lk(v_j, \Psi)$. Since p_j is not a vertex and the length of the edges of the link are $\pi/3$, we must have that $d(v'_j, p_j) > \pi$ in the link, and moreover, both vertices of e_j are distance $\geq \pi$ from v'_j. But now the rotation γ'_0 gives points p'_j in $lk(v_j, \Psi)$ which are vertices of the edge e_j, and hence has distance at least π from v'_j. This implies that γ' is locally geodesic at v_j for $j = 1$ and $j = n$. Hence γ' is a closed local geodesic in Ψ. But γ' is also an edge path, so by our previous remarks, we must have $\ell(\gamma) = \ell(\gamma') \geq 2\pi$.

This exhausts all possibilities for γ, so we see that any local geodesic loop must have length $\geq 2\pi$, implying Ψ is CAT(1). $\qquad\square$

6. The Moussong metric

In this section, we complete the proof of Theorem A. Our key lemma is the following, which we spend the section proving.

Lemma 6.1. *If G_Γ is finite with Γ connected and not type $A_4(3)$, then $\widehat{\Theta}_\Gamma$ is CAT(1) under the Moussong metric.*

We note that the case where Γ is Coxeter is well known, since $\widehat{\Theta}_\Gamma$ is the Coxeter complex, which, under the Moussong metric, is isometric to a sphere with its usual round metric. It remains to show the Lemma for non-Coxeter Γ. Our methods for proving the Lemma are case-specific, and as of yet we are unable to treat the $A_4(3)$ Shephard group. We hope to find a unified presentation in the future, hopefully one that includes $A_4(3)$.

We note here that, assuming the lemma, the proof of Theorem A follows quickly:

Proof of Theorem A (assuming Lemma 6.1). Let Γ be an extended Coxeter diagram with no subdiagram of the form $A_4(3)$ and with $\mathcal{S}^f = \mathcal{S}^{fs}$. By Proposition 3.8, the link of a vertex v_T for $T \in \mathcal{S}^f$ in the local development decomposes as $lk(v_T, F_T) * \widehat{\Theta}_{\Gamma(T)}$. Suppose $\Gamma(T)$ is connected. Since T generates a finite Shephard group and $\Gamma(T)$ is not $A_4(3)$, Lemma 6.1 implies $\widehat{\Theta}_{\Gamma(T)}$ is CAT(1). If $\Gamma(T)$ is the disjoint union $\Gamma_1 \sqcup \ldots \sqcup \Gamma_n$ of connected subdiagrams of Γ, then by Corollary 3.9, we know that $\widehat{\Theta}_{\Gamma(T)}$ is isometric to the spherical join $\widehat{\Theta}_{\Gamma_1} * \ldots * \widehat{\Theta}_{\Gamma_n}$. By assumption, no Γ_i is $A_4(3)$, so each $\widehat{\Theta}_{\Gamma_n}$ is CAT(1), and hence so is their spherical join $\widehat{\Theta}_{\Gamma(T)}$.

Let $K_0 = |(\mathcal{S}^f_{>\varnothing})'|$. We identify K_0 with a subspace of K; namely, it is easy to verify that $lk(v_\varnothing, F_\varnothing) \cong K_0$ as simplicial complexes. We put the subspace metric on K_0 coming from this identification. It is also straightforward to verify that $lk(v_T, K_0) \cong lk(v_T, F_T)$. Thus by [CD95, Lem. 4.4.1], $lk(v_T, F_T)$ is CAT(1). Therefore the orthogonal join of $\widehat{\Theta}_{\Gamma(T)}$ and $lk(v_T, F_T)$ is CAT(1). Since all vertices in the local development are a translate of some v_T, this means the local developments are nonpositively curved. Since $\mathcal{G}$ is a simple complex of groups over a simply connected fundamental domain, it follows that $\mathcal{G}$ has CAT(0) development Θ. By the definition of $\mathcal{G}$, G acts properly (all stabilizers are conjugates of the finite parabolics of G) and cocompactly (the fundamental domain is the compact space K) on Θ, and hence G is CAT(0). $\qquad\square$

6.1. **The 2-generator Shephard groups.** We begin with the 2-generator, or dihedral, Shephard groups $\Gamma = I_2(p, m, q)$. Let $\mathcal{P}$ be the regular complex polygon

associated to Γ. By [Sun75], the "girth" of $\mathcal{P}$ is m. In said article, the polygon (which is notated "$p\{m\}q$") is represented as a "hypergraph", and "girth of the polygon" means the combinatorial girth of this hypergraph, i.e., the minimal number of edges in a nontrivial closed loop. Since our complex $\widehat{\Theta}$ is the incidence graph of this polytope (and hence the corresponding hypergraph), it follows that the (combinatorial) girth of $\widehat{\Theta}$ is $2m$. Since the length of an edge is π/m, it follows that any non-trivial loop in $\widehat{\Theta}$ has length at least 2π, and hence $\widehat{\Theta}$ is CAT(1).

Example 6.2. Consider $\Gamma = I_2(p, 4, 2)$ for any $p \geq 2$. The associated complex $\widehat{\Theta}_\Gamma$ is isomorphic to the barycentric subdivision of $K_{p,p}$. See [Cox75, §4.8] for more details. Depending on which non-starry regular complex polytope for Γ we prefer (since it is not self-dual), the vertices of $K_{p,p}$ come from the vertices (or edges) of the complex polytope for Γ, and the barycenters of edges of $K_{p,p}$ come from the edges (resp. vertices) of the polytope. Some diagrams for low values of p are given in Figure 14. Black dots are the vertices (or edges) of the polytope for Γ, and white dots are edges (resp. vertices) of the polytope. There is an edge between two dots if the respective faces are nested.
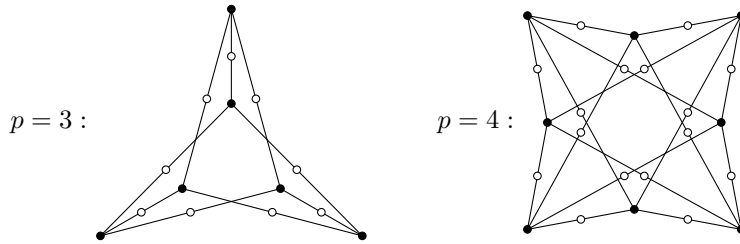

$p = 3:$ $p = 4:$

FIGURE 14. $\widehat{\Theta}(I_2(p, 4, 2))$

Example 6.3. Let $\Gamma = I_2(3, 3, 3)$. Then $\widehat{\Theta}_\Gamma$, shown in Figure 15, is the incidence graph of the *Mobius-Kantor configuration* 8_3. See [Cox75, §4.8] for more details. The figure given below for $\widehat{\Theta}_\Gamma$ is from [Orl90, Fig. 3]. Here, the black dots represent vertices of this configuration, and the white dots are the lines (or vice versa, since the polytope is self-dual).
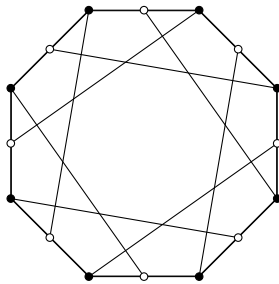

FIGURE 15. $\widehat{\Theta}(I_2(3, 3, 3))$

6.2. The groups $B_n(p, 2)$. The Shephard groups $G = G_\Gamma$ for $\Gamma = B_n(p, 2)$ are unique among the other finite Shephard groups, as they make up the only infinite family of finite non-Coxeter Shephard groups. However, this family becomes easy to deal with because of the following two lemmas.

Lemma 6.4. [Cox75, §12.2, p. 118] *Let γ_n^p denote the (unique non-starry) regular complex polytope with symmetry group G_Γ for $\Gamma = B_n(p, 2)$ (sometimes γ_n^p is called a* generalized n-cube). *Then,*

$$\gamma_n^p = \prod^n \gamma_1^p, \tag{6.1}$$

where $\prod^n \gamma_1^p$ denotes an n-fold direct product of γ_1^p.

Lemma 6.5. *The barycentric subdivision of the n-fold spherical join of a point has top dimensional cells isometric to the simplex of shape B_n.*

Proof. Let $Q = [0, 1]^n$ be the n-cube with its standard cellulation. The vertex link $lk(0, Q)$ is an "all-right simplex" (i.e., a spherical simplex with all edge lengths equal to $\pi/2$). This is the n-fold spherical join of one point (see [BH13, Def. I.5.13] for the definition of spherical join). Let Q' be the barycentric subdivision of Q (see [BH13, Def. I.7.42] for the definition of the barycentric subdivision of metrized cells). Then $lk(v, Q')$ is (simplicially) isometric to $lk(v, Q)'$, the latter of which is the barycentric subdivision of the all-right simplex, and hence the barycentric subdivision of the n-fold spherical join of a point.

Let v be the vertex of Q' coming from the top dimensional face of Q. The symmetry group of Q is the type B_n Coxeter group and thus $lk(v, Q')$ is isometric to the Coxeter complex for the B_n Coxeter group. In particular, the top dimensional cells of $lk(v, Q')$ are simplices of shape B_n. It follows easily from the definition of the barycentric subdivision that the top dimensional cells of $lk(0, Q')$ are isometric to the top dimensional cells of $lk(v, Q')$; the result follows. $\square$

From these, we can show

Proposition 6.6. *For all n and p, $\widehat{\Theta}(B_n(p, 2))$ is isometric to the barycentric subdivision of the n-fold spherical join of a set of p points, and hence is $\mathrm{CAT}(1)$.*

Proof. First, we recall basic constructions relating to posets. The (upper) cone $c\mathcal{P}$ on a poset $\mathcal{P}$ is the poset obtained by adding an element $1_\mathcal{P}$ (called the *cone point*) to $\mathcal{P}$ which we declare maximal in the order on $\mathcal{P}$. The derived complex $\mathcal{P}'$ is the set of all linearly ordered subsets (or "chains") of $\mathcal{P}$, ordered by inclusion. The derived complex $\mathcal{P}'$ is always an abstract simplicial complex, and we denote its geometric realization by $|\mathcal{P}'|$. The product $\mathcal{P} \times \mathcal{Q}$ is the poset whose underlying set is usual set-theoretic product $\mathcal{P} \times \mathcal{Q}$ with order $(p, q) \leq (p', q')$ if and only if $p \leq p'$ and $q \leq q'$. The join $\mathcal{P} * \mathcal{Q}$ is the poset $(c\mathcal{P} \times c\mathcal{Q}) \setminus \{(1_\mathcal{P}, 1_\mathcal{Q})\}$. We note that there is an order-preserving isomorphism $c(\mathcal{P} * \mathcal{Q}) \cong c\mathcal{P} \times c\mathcal{Q}$ which maps the cone point of $c(\mathcal{P} * \mathcal{Q})$ to $(1_\mathcal{P}, 1_\mathcal{Q})$.

We now turn to the setting of $B_n(p, 2)$. Let $(\gamma_n^p)_{prop}$ denote the poset of proper faces of γ_n^p (i.e., all faces except $\varnothing$ and $\mathbb{C}^n$), and let $(\gamma_n^p)_{>\varnothing}$ denote the set of non-empty faces of γ_n^p (but including $\mathbb{C}^n$). The poset $(\gamma_n^p)_{>\varnothing}$ is the cone on $(\gamma_n^p)_{prop}$ with cone point $\mathbb{C}^n$. By the comment after Definition 2.12, there is a poset isomorphism

$$\prod^n (\gamma_1^p)_{>\varnothing} \cong \left(\prod^n \gamma_1^p \right)_{>\varnothing},$$

where the product on the left hand side is the poset product, and that on the right is the polytope product. Thus by Lemma 6.4 we have

$$\prod_{n} c(\gamma_1^p)_{prop} = \prod_{n} (\gamma_1^p)_{>\varnothing} \cong (\gamma_n^p)_{>\varnothing} = c(\gamma_n^p)_{prop}.$$

And by our previous comments about products and joins of posets, we see that this gives the poset isomorphism

$$c\left(\overset{n}{\ast}(\gamma_1^p)_{prop} \right) \cong c(\gamma_n^p)_{prop},$$

where $\ast^n$ is the n-fold spherical join. In particular, the cone points are preserved in this map, so

$$\overset{n}{\ast}(\gamma_1^p)_{prop} \cong (\gamma_n^p)_{prop}.$$

Thus by taking geometric realizations, we have a simplicial homeomorphism

$$\left| \left(\overset{n}{\ast}(\gamma_1^p)_{prop} \right)' \right| \cong |(\gamma_n^p)'_{prop}|.$$

By our definitions, $\widehat{\Theta} \cong |(\gamma_n^p)'_{prop}|$. Moreover, the diagram $B_1(p, 2)$ is a single vertex labeled p; its regular complex polytope γ_1^p is a copy of $\mathbb{C}$ with p distingished points as vertices, so the poset $(\gamma_1^p)_{prop}$ is a set of p points, none of which are comparable. Thus $\ast^n(\gamma_1^p)_{prop}$ is the poset of cells of the n-fold spherical join of a set of p points, which we will call Δ, and $\left(\ast^n(\gamma_1^p)_{prop} \right)'$ is the poset of cells of the barycentric subdivision Δ' of Δ. In other words, there is a simplicial homeomorphism

$$\Delta' \cong \widehat{\Theta}(B_n(p, 2)).$$

The Moussong metric declares the top dimensional simplices of $\widehat{\Theta}(B_n(p, 2))$ to be simplices of type B_n. Lemma 6.5 implies that the top dimensional cells of Δ' are also simplices of type B_n; hence, this map is a simplicial isometry. Since Δ is CAT(1) (it is a spherical join of CAT(1) spaces), then Δ' is CAT(1) [BH13, Lem. I.7.48], and thus so is $\widehat{\Theta}(B_n(p, 2))$. $\qquad\square$

6.3. **The group $A_3(3)$.** We now consider $\widehat{\Theta} = \widehat{\Theta}(A_3(3))$. Throughout this section, let $\mathcal{H}$ denote the regular complex polytope associated to $A_3(3)$. Our arguments rely on the combinatorial structure of this polytope—see Appendix A for more details.

Here we utilize Charney's combinatorial CAT(1) criteria (CCCC) that we introduced in Section 5. Note that $\widehat{\Theta}$ carries a natural marked A_3 simplicial complex structure: we call a vertex of $\widehat{\Theta}$ type $\hat{a}$ if it is a vertex of $\mathcal{H}$, type $\hat{b}$ if it is an edge of $\mathcal{H}$, and type $\hat{c}$ if it is a face of $\mathcal{H}$. We also note that the canonical metric for marked A_3 simplicial complexes agrees with the Moussong metric on $\widehat{\Theta}$; both place an angle of $\pi/3$ at the vertices of type $\hat{a}$ and $\hat{c}$, and an angle of $\pi/2$ at the vertices of type $\hat{b}$.

Proposition 6.7. *Under the above marking, $\widehat{\Theta}(A_3(3))$ satisfies CCCC, and is therefore CAT(1) under its canonical metric (and hence the Moussong metric).*

Proof. The links of vertices of type $\hat{a}$ and $\hat{c}$ are isomorphic to $\widehat{\Theta}(I_2(3, 3, 3))$ which has girth 6, and the link of a vertex of type $\hat{b}$ is isomorphic to $\widehat{\Theta}(I_2(3, 2, 3))$, which by Corollary 3.9 is isomorphic to $K_{3,3}$.

It remains to show that $\widehat{\Theta}$ satisfies condition (3) of the criteria.

Path (i): Translating to $\mathcal{H}$, this setup is equivalent to taking two faces of $\mathcal{H}$ (corresponding to the vertices of type $\hat{c}$) which intersect in two distinct vertices of $\mathcal{H}$ (the vertices of type $\hat{a}$). By Proposition A.2, these faces must intersect along an edge containing these two vertices. This edge in the polytope corresponds to a vertex of type $\hat{b}$ which is joined to each vertex of γ, and each of the triangles formed are filled as in Figure 3(i).

Path (ii): Path (ii.a) says we have three edges in the complex polytope (vertices of type $\hat{b}$) which pairwise intersect at three distinct vertices of the polytope (vertices of type $\hat{a}$). By Proposition A.1, these edges must be contained in some common face (a vertex of type $\hat{c}$), giving Figure 3(ii.a). The (ii.b) case follows from passing to the dual polytope.

Thus we conclude that $\widehat{\Theta}(A_3(3))$ satisfies CCCC, and is therefore CAT(1). $\qquad\square$

6.4. **The group** $B_3(2,3)$**.** By [Cox75, Sec. 12.4], we know $G_{B_3(2,3)} \cong (\mathbb{Z}/2\mathbb{Z}) \times G_{A_3(3)}$. This manifests on the level of complexes; $\widehat{\Theta}(B_3(2,3))$ is a subdivision of $\widehat{\Theta}(A_3(3))$. Specifically, doubling the fundamental domain for $\widehat{\Theta}(B_3(2,3))$ along the face corresponding to the order-2 generator is isometric to the fundamental domain for $\widehat{\Theta}(A_3(3))$. This is illustrated in Figure 16, where the fundamental domain for $B_3(2,3)$ is outlined in bold, with the corresponding stabilizers labeling the edges. The vertices of $\Delta_{A_3(3)}$ are labeled according to the convention used in the previous section. Since this subdivision respects the metric on $\widehat{\Theta}(A_3(3))$, it follows that $\widehat{\Theta}(B_3(2,3))$ is CAT(1).
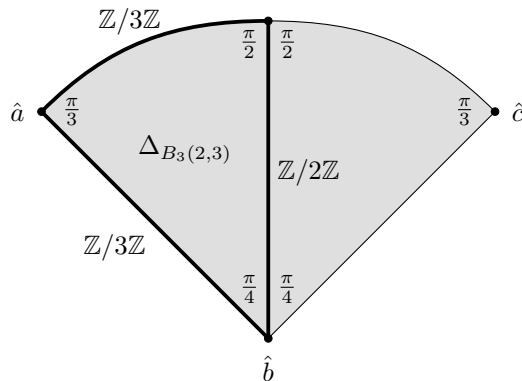

FIGURE 16. $\Delta_{B_3(2,3)}$ as a subspace of $\Delta_{A_3(3)}$

This exhausts the list of finite Shephard groups with connected diagram (excluding $A_4(3)$), thus completing the proof of Lemma 6.1, and by extension, Theorem A.

APPENDIX A. COMBINATORICS OF THE HESSIAN POLYHEDRON

In this section we provide explanations for many technical lemmas regarding the $A_3(3)$ Shephard group and lay out explicit data utilized in the computations therein. Throughout, let $\mathcal{H}$ denote the standard Hessian polyhedron, that is, the non-starry regular complex polytope with diagram $A_3(3)$. We begin by recalling basic facts about the facets of $\mathcal{H}$. The following can be found in [Cox75, Ch. 12.3].

The 27 vertices of $\mathcal{H}$ are given by

$$(0, \omega^i, -\omega^j) \qquad (-\omega^j, 0, \omega^i) \qquad (\omega^i, -\omega^j, 0)$$

where $\omega = \exp(2\pi\sqrt{-1}/3)$ and $i, j = 1, 2, 3$. As in Coxeter, we find it convenient to use the shorthand

$$0ij \qquad j0i \qquad ij0$$

for $i, j = 1, 2, 3$ to represent the respective vertices. (For example, 013 is the vertex $(0, \omega^1, -\omega^3)$.) The generators of the $A_3(3)$ Shephard group have a nice description as permutations of these vertices, which is listed in Table 6.

$a = (101\ 201\ 301)(102\ 202\ 302)(103\ 203\ 303)(110\ 210\ 310)(120\ 220\ 320)(130\ 230\ 330)$

$b = (012\ 230\ 103)(013\ 102\ 320)(021\ 203\ 130)(023\ 310\ 201)(031\ 120\ 302)(032\ 301\ 210)$

$c = (011\ 012\ 013)(021\ 022\ 023)(031\ 032\ 033)(101\ 102\ 103)(201\ 202\ 203)(301\ 302\ 303).$

TABLE 6. Permutation representation of $A_3(3)$

There are 72 edges of $\mathcal{H}$. Two vertices lay on a common edge if and only if their symbols agree in an even number of positions, meaning they agree in two positions or no position. For example, 013 and 023 share an edge, while 013 and 022 do not. Thus it is straightforward to verify if two given points share an edge in $\mathcal{H}$ despite the (somewhat) large number of edges. In addition, since "edges" of a complex polytope are (affine) complex lines, and two points determine a unique line, two vertices share *at most* one edge.

We next describe the (2-)faces. Since $\mathcal{H}$ is self-dual, it has 27 faces. One of the faces has vertices

$$012, 021, 103, 203, 303, 130, 230, 330,$$

which is the orbit of 012 under the subgroup generated by a and b. The other faces are the translates of this face by elements of the symmetry group. We provide the vertices of these faces here for posterity (including the face mentioned above) in Table 7. We note that the list of vertices is sufficient to determine the entire face; an edge is included in a face if and only if each of its vertices are. That is to say that if three vertices in a face span an edge of $\mathcal{H}$, then this edge is part of said face.

We conclude with two further observations which can be derived from examining Table 7. First,

Proposition A.1. *If E_1, E_2, and E_3 are distinct edges of $\mathcal{H}$ which pairwise intersect non-trivially but have trivial total intersection, then there is a face F of $\mathcal{H}$ containing E_1, E_2, and E_3.*

This essentially says there are no "empty triangles" in $\mathcal{H}$. The following says that the intersection of faces is well behaved in a certain way.

Proposition A.2. *If F_1 and F_2 are distinct faces of $\mathcal{H}$, then $F_1 \cap F_2$ is either empty, a single vertex, or a single edge.*

Again, both of these may be proven "by hand" (or more conveniently, by computer) using Table 7 and the previously stated characterization of the edges of $\mathcal{H}$.

1 : 012 021 103 203 303 130 230 330   2 : 302 022 023 110 301 303 230 021

3 : 202 022 310 201 023 130 021 203   4 : 320 022 101 310 032 203 012 330

5 : 102 022 101 023 210 021 103 330   6 : 011 023 102 202 302 130 230 330

7 : 220 022 210 032 301 012 230 103   8 : 120 022 032 201 110 130 303 012

9 : 031 022 310 110 210 303 103 203   10 : 011 310 320 202 031 021 103 330

11 : 320 023 102 310 033 201 013 330   12 : 220 023 210 033 302 013 230 101

13 : 120 023 033 202 110 130 301 013   14 : 032 023 310 110 210 301 101 201

15 : 220 021 210 031 303 011 230 102   16 : 120 021 031 203 110 130 302 011

17 : 033 021 310 110 210 302 102 202   18 : 220 310 032 033 031 102 103 101

19 : 220 203 032 120 303 011 103 320   20 : 033 203 201 110 032 031 320 202

21 : 220 201 301 033 120 320 012 101   22 : 220 130 301 302 303 011 012 013

23 : 033 303 301 210 032 031 120 302   24 : 320 230 101 102 103 011 012 013

25 : 220 202 302 031 120 320 013 102   26 : 120 330 201 202 203 011 012 013

27 : 013 022 101 201 301 130 230 330

TABLE 7. The 27 faces of $\mathcal{H}$

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