THE $K(\pi, 1)$ CONJECTURE AND ACYLINDRICAL HYPERBOLICITY FOR RELATIVELY EXTRA-LARGE ARTIN GROUPS

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Abstract. Let A_Γ be an Artin group with defining graph Γ. We introduce the notion of A_Γ being extra-large relative to a family of arbitrary parabolic subgroups. This generalizes a related notion of A_Γ being extra-large relative to two parabolic subgroups, one of which is always large type. Under this new condition, we show that A_Γ satisfies the $K(\pi, 1)$ conjecture whenever each of the distinguished subgroups do. In addition, we show that A_Γ is acylindrically hyperbolic under only mild conditions.

Let Γ be a finite simplicial graph whose edges are labeled with (finite) integers, each at least 2. For vertices s, t of Γ connected by an edge, let $m(s, t)$ denote the label of the edge between s and t. Let $S = \mathrm{Vert}(\Gamma)$. Since Γ is simplicial, we use the convention that an edge of Γ is the same as an unordered pair $\{s, t\}$ of vertices of Γ. The Artin group defined by Γ is

$$A_\Gamma = \langle\, S \mid \mathrm{prod}(s, t; m(s, t)) = \mathrm{prod}(t, s; m(s, t)) \text{ for } \{s, t\} \text{ an edge of } \Gamma \,\rangle,$$

where $\mathrm{prod}(a, b; n)$ is the alternating word in a and b starting with a of length n (eg, $aba\ldots$). We call the pair (A, S) an *Artin-Tits system*.

There is a Coxeter group also naturally associated with this defining graph; namely,

$$W_\Gamma = \langle\, S \mid (st)^{m(s,t)} = 1 \text{ for } \{s, t\} \text{ an edge of } \Gamma,\ s^2 = 1 \text{ for } s \in S \,\rangle.$$

It is well known that there is a natural surjective homomorphism $A_\Gamma \to W_\Gamma$ induced by the identity map on S. Recall that if W_Γ is finite, then we call W_Γ *spherical* and call A_Γ *spherical-type*. In this case, we may sometimes refer to Γ itself as *spherical-type*.

By van der Lek [vdL83], if Γ' is a full (or "induced") subgraph of Γ, then the natural map from the Artin group $A_{\Gamma'}$ to A_Γ is an injection. (Recall that a subgraph Γ' of Γ is called full if for any pair of vertices v, w of Γ' which span an edge $\{v, w\}$ in Γ, we also have that $\{v, w\}$ is an edge of Γ'.) We call such a subgroup of A_Γ a *(standard) parabolic subgroup*. Sometimes, if $T = \mathrm{Vert}(\Gamma')$, we write A_T for $A_{\Gamma'}$.

It is also well known that the Artin group A_Γ is the fundamental group of a space $N(W)$ which is the quotient of a complement of a certain complexified hyperplane arrangement by a natural W_Γ-action. (See [Par14] for more details.) The long-standing $K(\pi, 1)$ conjecture states that $N(W)$ is aspherical (ie, has contractible universal cover). Currently, the $K(\pi, 1)$ conjecture is known to be true when

(1) A_Γ is spherical-type (in [Del72]),
(2) A_Γ is affine-type (meaning W_Γ has a finite-index subgroup which acts properly by isometries on a Euclidean space) (proven in general in [PS21]),

(3) if Γ' is a full spherical-type subgraph of Γ then $|\mathrm{Vert}(\Gamma')| = 2$ (in which case A_Γ is called 2-dimensional) (in [CD95]), or more generally, if A_Γ is locally reducible (in [Cha00]),
(4) if every full complete subgraph of Γ is spherical-type (in which case A_Γ is called FC type) (also in [CD95]), and
(5) some combination criteria are satisfied, including results by Godelle and Paris [GP12] and Ellis and Sköldberg [ES10].

We present a new criterion based on the following familiar condition: an Artin group A_Γ is *extra-large type* if every edge of Γ has label at least 4. In this case, A_Γ is 2-dimensional, and thus satisfies the $K(\pi, 1)$ conjecture. In [Juh18], the following condition is introduced. Let $H = A_{\Gamma'}$ be a standard parabolic subgroup of A (with $\Gamma' \subseteq \Gamma$ a full subgraph). Then A is *extra-large relative to* H (or Γ'*-relatively extra-large*) if

(1) for every edge $\{s, t\}$ of Γ with $s \in \Gamma'$ and $t \notin \Gamma'$, we have $m(s, t) \geq 4$, and
(2) for every edge $\{t, t'\}$ of Γ with $t, t' \notin \Gamma'$, we have $m(t, t') \geq 3$.

It is then shown that A_Γ satisfies the word problem or $K(\pi, 1)$ conjecture whenever H does. It is in this spirit that we make the following generalization.

Let $\{\Gamma_i\}$ be a finite family consisting of disjoint, non-empty full subgraphs of Γ with vertex sets $S = \mathrm{Vert}(\Gamma)$ and $S_i = \mathrm{Vert}(\Gamma_i)$. Suppose also that $S = \bigcup S_i$. In direct analogy to the relatively extra-large condition, we consider

(REL) Every edge of Γ between Γ_i and Γ_j for some $i \neq j$ has label at least 4.

If this condition is satisfied, we say that A_Γ is $\{\Gamma_i\}$*-relatively extra-large.* We establish the following theorem regarding such Artin groups.

Theorem. *Suppose A_Γ is $\{\Gamma_i\}$-relatively extra-large. Then A_Γ satisfies the $K(\pi, 1)$ conjecture if and only if each A_{Γ_i} does.*

In fact, a somewhat stronger fact can be established using our methods. Instead of (REL), consider

(REL′) If e is an edge of Γ between Γ_i and Γ_j for some $i \neq j$ and e shares a vertex with a distinct edge between Γ_i and Γ_k for some $i \neq k$, then e has label at least 4.

Specifically, this allows edges which are isolated among those edges between the subgraphs in the family $\{\Gamma_i\}$ to have label 2 or 3. We show

Theorem A. *Suppose* Γ *and* $\{\Gamma_i\}$ *satisfy* (REL′). *Then A_Γ satisfies the $K(\pi, 1)$ conjecture if and only if each A_{Γ_i} does.*

In addition to this, we are able to show under mild hypotheses that Artin groups satisfying (REL′) are also acylindrically hyperbolic. Acylindrical hyperbolicity is a property of interest for many groups, including Artin groups. Some of the classes for which acylindrical hyperbolicity is known for include

(1) right-angled Artin groups ($m(s, t) = 2$ for each edge of Γ) which are not cyclic or a direct product of non-trivial subgroups (in [Osi16]),
(2) spherical-type Artin groups (in [CW17]),
(3) type FC Artin groups whose defining graph has diameter at least 3 (in [CM19]),
(4) extra-extra-large type Artin groups (meaning $m(s, t) \geq 5$ for each edge $\{s, t\}$ of the defining graph) of rank at least 3 (in [Hae19]),

(5) Artin groups A_Γ such that Γ is not a join of two subgraphs Γ_1, Γ_2 (in [CMW19]),
(6) affine-type Artin groups (in [Cal20]),
(7) 2-dimensional Artin groups of hyperbolic type (meaning the associated Coxeter group is hyperbolic) (in [MP19]), and
(8) 2-dimensional Artin groups (in [Vas21]).

We show acylindrical hyperbolicity in our setting as well:

Theorem B. *Suppose A_Γ and $\{\Gamma_i\}_{i=1}^n$, $n \geq 2$ satisfy* (REL$'$). *In addition, assume $|\mathrm{Vert}(\Gamma)| \geq 3$ and not all edges between the family $\{\Gamma_i\}$ have label 2. Then A_Γ is acylindrically hyperbolic.*

We note that the conditions in Theorem A include the original relatively extra-large condition of Juhász as a special case. Suppose A_Γ is Γ'-relatively extra-large (in the sense of [Juh18]). Let Γ'' be the full subgraph on the vertices of Γ which are not in Γ'. Then A_Γ is $\{\Gamma', \Gamma''\}$-relatively extra-large in our sense. The condition (2) in the definition of Γ-relatively extra-large is equivalent to requiring that $A_{\Gamma''}$ be large type (ie, all edge labels are at least 3). Then $A_{\Gamma''}$ satisfies the $K(\pi, 1)$ conjecture as $A_{\Gamma''}$ is 2-dimensional. Thus according to our result, A_Γ satisfies the $K(\pi, 1)$ conjecture if and only if $A_{\Gamma'}$ does.

Our Theorems include many new examples for which the $K(\pi, 1)$ conjecture and/or acylindrical hyperbolicity was not previously known. As one example, consider two graphs Γ_1, Γ_2 of type $\widetilde{C}_3$ (see Figure 1). These defining graphs generate

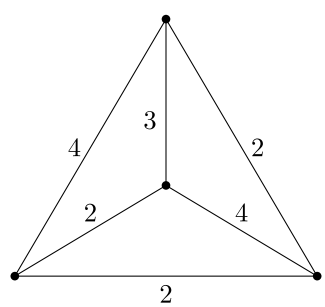


FIGURE 1. A defining graph of type $\widetilde{C}_3$

an affine Artin group, and thus satisfy the $K(\pi, 1)$ conjecture. Then, let Γ be the join of Γ_1 and Γ_2 with each of the new edges labeled by 4 (or greater). It is quickly checked that A_Γ satisfies none of the previously listed conditions. But, Γ and $\{\Gamma_1, \Gamma_2\}$ satisfy (REL$'$), and each A_{Γ_i} satisfies the $K(\pi, 1)$ conjecture, so A_Γ does. In addition, none of the edges between Γ_1 and Γ_2 are labeled 2, so A_Γ is acylindrically hyperbolic. More generally, if a clique Γ with at least 3 vertices is extra-large relative to a family $\{\Gamma_i\}$, then A_Γ is acylindrically hyperbolic, and if each Γ_i satisfies the $K(\pi, 1)$ conjecture, then A_Γ does as well.

We also note that our methodology for proving Theorem A differs from Juhász' original work, allowing us to drop his condition (2) and treat more general defining graphs. This also allows us to easily prove acylindrical hyperbolicity. We hope that this method may be adapted for other similar restrictions on A_Γ. Namely, our strategy for proving Theorem A is as follows. In Section 1, we construct a simplicial complex as a variation of the usual Deligne complex. We show the complex is CAT(0), hence contractible, in Section 2. Then in Section 3, we show that this complex is homotopy equivalent to the universal cover of $N(W)$ by a result of Godelle and Paris [GP12]. In Section 4, we prove Theorem B using recent results of Vaskou [Vas21].

We would also like to note that the conditions (REL) and (REL′) can be naturally relaxed to allow edges with label at least 3, which would define a *relatively large type* condition. This case is also currently of interest to the author; however, it is somewhat more complicated than the current case of consideration.

The author would like to extend great thanks to Mike Davis and Jingyin Huang for their helpful comments and advice given through the writing of this paper.

1. The Deligne-like complex

Before we define our complex, we wish to establish a lemma in Artin groups similar to a well-known property of cosets of standard parabolic subgroups of Coxeter groups. We include a proof for the reader's convenience. We make heavy use of this result in the subsequent sections.

Lemma 1.1. *Suppose (A, S) is an Artin-Tits system, $\alpha, \alpha' \in A$, and $T, T' \subseteq S$. Then if $\alpha A_T \subseteq \alpha' A_{T'}$, we have $\alpha^{-1}\alpha' \in A_{T'}$ and $T \subseteq T'$.*

Proof. Let w and w' be the image of α and α', respectively, under the quotient homomorphism $A_\Gamma \to W_\Gamma$. The inclusion $\alpha A_T \subseteq \alpha' A_{T'}$ is preserved under the quotient map, giving us the relation $wW_T \subseteq w'W_{T'}$ in W_Γ. So, by [Bou02, Ch. IV §8 Th. 2(iii)], we have $T \subseteq T'$ as subsets of W_Γ. Since the quotient map is bijective on the generators, this gives $T \subseteq T'$ viewed in A_Γ.

To see that α and α' must be in the same $A_{T'}$-coset, note that $\alpha A_T \subseteq \alpha A_{T'}$ as well as $\alpha A_T \subseteq \alpha' A_{T'}$, so $\varnothing \neq \alpha A_T \subseteq \alpha A_{T'} \cap \alpha' A_{T'}$. Since cosets partition the group and these cosets have non-empty intersection, they must be the same. □

We also briefly give a restatement of a result of van der Lek.

Lemma 1.2. *If (A, S) is an Artin-Tits system and $s \in S$, then s cannot be written as a product of the elements of $S \setminus \{s\}$.*

Proof. By van der Lek's thesis [vdL83],

$$A_{\{s\}} \cap A_{S\setminus\{s\}} \cong A_{\{s\}\cap S\setminus\{s\}} = A_\varnothing = 1.$$

Thus in particular, $s \notin A_{S\setminus\{s\}}$. Since $A_{S\setminus\{s\}}$ is the collection of all possible products of the generators $S \setminus \{s\}$, the result follows. □

1.1. Definition of the complex.

Through the rest of the paper, we let $A = A_\Gamma$ be an Artin group such that Γ and $\{\Gamma_i\}$ satisfy (REL′), with $S_i = \mathrm{Vert}(\Gamma_i)$ and $A_i = A_{\Gamma_i}$.

We now introduce a simplicial complex based on our distinguished subgroups A_i of A analogous to the Deligne complex. To do this, we mimic the construction of the Deligne complex in [CD95], but replace the poset of spherical generating sets with the following set.

Definition 1.3. Let $\mathcal{S}^\ell$ be the set of all $T \subseteq S$ satisfying either

(1) $T = \varnothing$ (in which case $A_T = 1$, the trivial subgroup of A),
(2) $T = S_i$,

(3) $T = \{s_i, s_j\}$ for vertices $s_i \in S_i$, $s_j \in S_j$ of an edge between Γ_i and Γ_j with $i \neq j$, or
(4) $T = \{s\}$ for a vertex s of an edge between Γ_i and Γ_j, $i \neq j$.

With this, we define

$$A\mathcal{S}^\ell = \{ \alpha A_T \,:\, \alpha \in A, T \in \mathcal{S}^\ell \},$$

and order these sets by inclusion. We then let X denote the geometric realization of the derived complex of $\mathcal{S}^\ell$ and $\hat{\Phi}$ denote the geometric realization of the derived complex of $A\mathcal{S}^\ell$ (recall that the derived complex of a poset is the set of chains in the poset ordered by inclusion of chains).

We will denote an n-simplex of $\hat{\Phi}$ by

$$[\alpha_0 A_{T_0},\, \alpha_1 A_{T_1},\, \ldots,\, \alpha_n A_{T_n}]$$

where $\alpha_0 A_{T_0} < \alpha_1 A_{T_1} < \cdots < \alpha_n A_{T_n}$ is a chain in $A\mathcal{S}^\ell$. We use similar notation for simplices of X. Notice that $\hat{\Phi}$ inherits a natural left action of A with fundamental domain isomorphic to X via the simplicial map induced by the set map $T \mapsto A_T$.

We note that if one replaces $\mathcal{S}^\ell$ by $\mathcal{S}^f$, the set of $T \subseteq S$ so that A_T is spherical-type, then the definition of the (modified) Deligne complex of Charney and Davis [CD95] is recovered. To further borrow their notation, we will let K denote the geometric realization of the derived complex of $\mathcal{S}^f$, let $A\mathcal{S}^f$ denote the cosets of A_T for $T \in \mathcal{S}^f$, and let $\Phi_M = \Phi_M(A_\Gamma)$ denote the geometric realization of the derived complex of $A\mathcal{S}^f$.

The rest of this section and the next is dedicated to showing that $\hat{\Phi}$ is CAT(0). First, we show that $\hat{\Phi}$ is simply connected, then endow it with a metric of non-positive curvature.

To show that $\hat{\Phi}$ is simply connected, we will use basic facts about complexes of groups. We will only need the fact that the action of A_Γ on $\hat{\Phi}$ has a complex of groups structure briefly, so we will summarize the basic argument here, and refer the reader to [Hae92] for more details on complexes of groups.

Lemma 1.4. *The complex $\hat{\Phi}$ is simply connected.*

Proof. The stabilizer of a vertex $[\alpha A_T]$ of $\hat{\Phi}$ is the subgroup $\alpha A_T \alpha^{-1}$ of A. Thus A acts on $\hat{\Phi}$ without inversion. The complex X is homeomorphic to the quotient $\hat{\Phi}/A$ via the simplicial map induced by $T \mapsto A_T$. In addition, X is simply connected, as $[\varnothing]$ is a cone point in X. This information determines a complex of groups [Hae92, Section 2.1], which we denote by $A(X)$. The edge maps are the usual inclusion maps $A_T \hookrightarrow A_{T'}$. Note that this complex is developable by definition.

Since X is simply connected, $\pi_1(A(X))$ is the colimit of the groups A_T along the inclusion maps [Hae92, Section 2.7], implying $\pi_1(A(X)) = A$. It follows that the classifying space of $A(X)$ is $BA(X) = \hat{\Phi} \times_A EA$ [Hae92, Prop. 3.2.3], and thus the universal cover is

$$\widetilde{BA}(X) = \hat{\Phi} \times EA,$$

which is homotopy equivalent to $\hat{\Phi}$. This shows that $\hat{\Phi}$ is simply connected. □

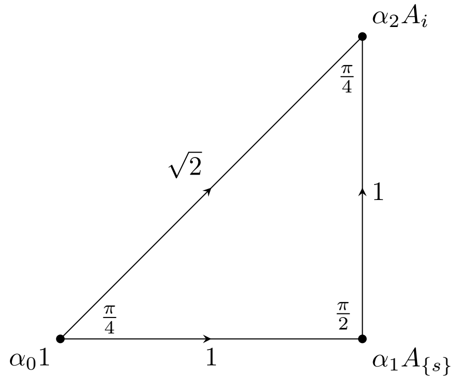
1.2. **The metric on $\hat{\Phi}$.** In order to put a metric on $\hat{\Phi}$, we first note the following

Lemma 1.5. *The complex $\hat{\Phi}$ is 2-dimensional.*

Proof. Suppose we have a 3-simplex $[\alpha_0 A_{T_0}, \alpha_1 A_{T_1}, \alpha_2 A_{T_2}, \alpha_3 A_{T_3}]$ of $\hat{\Phi}$. By Lemma 1.1, we then have a chain $T_0 < T_1 < T_2 < T_3$. In particular, $|T_2| \geq 2$. The only sets of $\mathcal{S}^\ell$ with cardinality at least 2 are either S_i for some i or an edge $\{s_i, s_j\}$. But in either case, there is no element of $\mathcal{S}^\ell$ containing T_2, a contradiction. □

As a consequence of the proof of the Lemma, there are only two kinds of top-dimensional simplices of $\hat{\Phi}$: the first is $[\alpha_0 1, \alpha_1 A_{\{s\}}, \alpha_2 A_i]$ for a vertex $s \in S_i$ of an edge between Γ_i and some Γ_j, and the second is $[\alpha_0 1, \alpha_1 A_{\{s_i\}}, \alpha_2 A_{\{s_i,s_j\}}]$ for $\{s_i, s_j\}$ an edge between Γ_i and Γ_j.

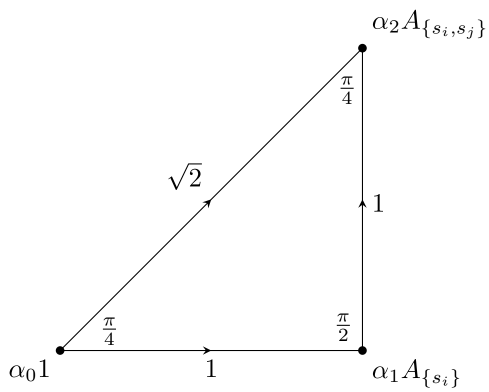
We now put a metric on the two kinds of 2-simplices of $\hat{\Phi}$. First, consider $[\alpha_0 1, \alpha_1 A_{\{s\}}, \alpha_2 A_i]$. We give this simplex the metric of a Euclidean isosceles right triangle with right angle at $\alpha_1 A_{\{s\}}$ and whose legs have length 1. Pictorially, we have



The arrows here denote the inclusion of the relevant groups.

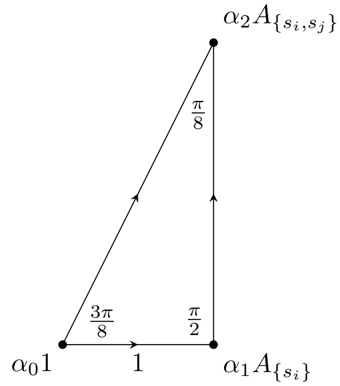
Now consider a simplex of the form $\Delta = [\alpha_0 1, \alpha_1 A_{\{s_i\}}, \alpha_2 A_{\{s_i,s_j\}}]$ for $e = \{s_i, s_j\}$ an edge between Γ_i and Γ_j.

1.2.1. *Case 1: A disjoint edge.* Suppose that e is disjoint from all other edges between any Γ_k and Γ_ℓ. We then put a similar metric on Δ as in the previous case; namely,



1.2.2. *Case 2: A non-disjoint edge.* Now suppose that e shares a vertex with some other edge between Γ_i and Γ_j. Then we still put the metric of a Euclidean right triangle on Δ, but it will no longer be isosceles. Specifically, the metric we put on

Δ still assigns a right angle to the vertex $\alpha_1 A_{\{s_i\}}$, but now places an angle of $3\pi/8$ to $\alpha_0 1$ and an angle of $\pi/8$ to $\alpha_2 A_{\{s_i,s_j\}}$. Moreover, importantly, the 1-simplex $[\alpha_0 1, \alpha_1 A_{\{s_i\}}]$ is given length 1. The diagram for this case is



In order to show this properly defines a piecewise Euclidean metric on $\hat{\Phi}$, we examine the gluings between adjacent simplices. We begin with a simplex of the form $\Delta = [\alpha_0 1, \alpha_1 A_{\{s\}}, \alpha_2 A_i]$. The only type of simplex Δ can be adjacent to which is not of the same type is one of the form $\Delta' = [\alpha_0 1, \alpha_1 A_{\{s\}}, \alpha_2' A_{\{s,t\}}]$ with $\{s,t\}$ an edge between Γ_i and Γ_j, and t a vertex of Γ_j. These simplices are glued only along the edge $[\alpha_0 1, \alpha_1 A_{\{s\}}]$, and within both simplices we have assigned this edge a length of 1.

Now consider $\Delta = [\alpha_0 1, \alpha_1 A_{\{s_i\}}, \alpha_2 A_{\{s_i,s_j\}}]$ for $\{s_i, s_j\}$ an edge between Γ_i and Γ_j. The case where Δ is adjacent to a simplex of the form $[\alpha_0 1, \alpha_1 A_{\{s\}}, \alpha_2 A_i]$ was covered above. So, consider an adjacent simplex of the form $[\alpha_0' 1, \alpha_1 A_{\{s_i\}}, \alpha_2 A_{\{s_i,s_j\}}]$ or $[\alpha_0 1, \alpha_1' A_{\{s_i'\}}, \alpha_2 A_{\{s_i,s_j\}}]$. In either case, the metric put on the simplices is the same as that of Δ as this metric only depended on the edge $\{s_i, s_j\}$, so there is no issue with the gluing.

It remains to check the simplices of the form $\Delta' = [\alpha_0 1, \alpha_1 A_{\{s_i\}}, \alpha_2 A_{\{s_i', s_k\}}]$ for an edge $\{s_i', s_j'\}$ and $s_k \in \Gamma_k$ for some $k \neq i$. By Lemma 1.1, since $\alpha_1 A_{\{s_i\}} \subseteq \alpha_2 A_{\{s_i', s_k\}}$, we have $\{s_i\} \subseteq \{s_i', s_k\}$, and since $s_k \in \Gamma_k$ we must have $s_i' = s_i$. Thus if this is to be a simplex distinct from Δ, we must have $s_k \neq s_j$, so $\{s_i, s_k\}$ and $\{s_i, s_j\}$ are both edges which are not distinct. Thus the metrics on Δ and Δ' are the same, so they may be glued as required.

2. Links

The purpose of this section is to show the following.

Proposition 2.1. *The complex $\hat{\Phi}$ (with the above metric) is* CAT(0) *(hence contractible).*

To do this, we compute the link at each relevant vertex of $\hat{\Phi}$ and show that the link condition is satisfied. Let us briefly recall the relevant definitions. (For more details, see [BH13].)

Definition 2.2 (Link of a vertex)**.** Let K be a polyhedral complex and v a vertex of K. Then the link of v in K, denoted $\mathrm{lk}_K(v)$, is the ε-sphere of K centered at v. We give the link a cell structure coming from the intersection of the sphere with the cell structure of K. The link is endowed with a natural spherical metric inherited from the ε-sphere.

In the case of the geometric realization of an abstract simplicial complex (such as $\hat{\Phi}$), we can give an explicit description of the link of a vertex using the underlying set. Let $[\alpha A_T]$ be a vertex of $\hat{\Phi}$ (so $\alpha A_T \in A\mathcal{S}^\ell$). Then the vertex set of $\mathrm{lk}_{\hat{\Phi}}([\alpha A_T])$ is

$$\{\, \alpha' A_{T'} \,:\, \alpha' A_{T'} \subseteq \alpha A_T \,\} \cup \{\, \alpha'' A_{T''} \,:\, \alpha'' A_{T''} \supseteq \alpha A_T \,\}.$$

But by Lemma 1.1, this is the same as the set

$$\{\, \alpha' A_{T'} \,:\, \alpha' A_{T'} \subseteq \alpha A_T \,\} \cup \{\, \alpha A_{T''} \,:\, T'' \supseteq T \,\}.$$

A collection of vertices $\alpha_0 A_{T_0} < \cdots < \alpha_j A_{T_j} < \alpha A_{T'_0} < \ldots \alpha A_{T'_k}$ span a $(j+k)$-simplex of $\mathrm{lk}_{\hat{\Phi}}([\alpha A_T])$ if and only if

$$[\alpha_0 A_{T_0} < \cdots < \alpha_j A_{T_j} < \alpha A_T < \alpha A_{T'_0} < \ldots \alpha A_{T'_k}]$$

is a $(j+k+1)$-simplex of $\hat{\Phi}$. In the case of $\hat{\Phi}$, we can say slightly more than this. Our complex $\hat{\Phi}$ is 2-dimensional, so the link of any vertex is 1-dimensional. Moreover, the link of a simplicial complex is itself a simplicial complex, so the link here is always a simplicial graph.

We can also explicitly describe the spherical metric on each link in $\hat{\Phi}$. If $[\alpha A_T]$ is a vertex of $\hat{\Phi}$ and $e = [\alpha_0 A_{T_0}, \alpha_1 A_{T_1}]$ is an edge of $\mathrm{lk}_{\hat{\Phi}}([\alpha A_T])$, then the length of e is the angle assigned above to the vertex corresponding to αA_T in the simplex of $\hat{\Phi}$ spanned by the vertices αA_T, $\alpha_0 A_{T_0}$, $\alpha_1 A_{T_1}$.

Definition 2.3. We say that a polyhedral complex K satisfies the *link condition* if for each vertex v of K, the link $\mathrm{lk}_K(v)$ is a CAT(1) space (under the induced spherical metric).

To show $\hat{\Phi}$ is CAT(0), we make use of the following criterion, proven in [BH13].

Lemma 2.4. *If K is a Euclidean polyhedral complex (meaning each cell of K has the metric of a Euclidean polytope) and K is simply connected, then K is* CAT(0) *if and only if it satisfies the link condition.*

Since our complex $\hat{\Phi}$ is 2-dimensional, to verify our links are CAT(1), we can use the following equivalent condition, also proven in [BH13].

Lemma 2.5. *A 2-dimensional Euclidean simplicial complex K satisfies the link condition if and only if for each vertex v of K, every embedded closed loop in* $\mathrm{lk}_K(v)$ *has length at least 2π.*

We now turn to examining the links of our complex in detail. Since each vertex of $\hat{\Phi}$ is a translate of one of the cosets A_T, it suffices to just compute the link at A_T for $T \in \mathcal{S}^\ell$.

2.1. **Case 1: $T = S_i$.** Let us first examine the link of A_i for fixed i. The vertex set of this link can be decomposed as

$$\{\, \alpha 1 \,:\, \alpha \in A_i \,\} \qquad \text{and} \qquad \{\, \alpha A_{\{s\}} \,:\, \alpha \in A_i, s \in S_i \,\}.$$

It is easily seen that there is no edge between any two vertices which are in the same set, meaning the link is a bipartite graph. By definition, we can only have an edge when $\alpha 1 \subseteq \alpha' A_{\{s\}}$, or in other words, when $\alpha \in \alpha' A_{\{s\}}$.

To show that the shortest embedded closed loop in A_i has length at least 2π, we claim that any embedded closed loop in $\mathrm{lk}_{\hat{\Phi}}(A_i)$ must have at least 8 edges. Since the link is a bipartite graph, we know the edge length of any cycle is even and at least 4. So, we only need to verify that there are no cycles of edge length 4 or 6.

Suppose we have a loop with 4 edges. Then by our discussion regarding the possible edges in the link, this loop must have the form

$$\begin{array}{ccc} & \alpha_1 1 & \\ \swarrow & & \searrow \\ \alpha' A_{\{s'\}} & & \alpha'' A_{\{s''\}} \\ \nwarrow & & \nearrow \\ & \alpha_2 1 & \end{array}$$

(The arrows correspond to inclusions; the paths we consider are not directed.) This gives us equations of the form

$$\alpha'(s')^{k_1} = \alpha_1 = \alpha''(s'')^{j_1}$$
$$\alpha'(s')^{k_2} = \alpha_2 = \alpha''(s'')^{j_2}.$$

Or rewriting,

$$(s'')^{j_1}(s')^{-k_1} = (\alpha'')^{-1}\alpha' = (s'')^{j_2}(s')^{-k_2},$$

implying

$$(s'')^{j_1-j_2} = (s')^{k_1-k_2}.$$

Since we're assuming the loop is embedded, $s' \neq s''$ (otherwise, two cosets of the same subgroup $A_{s'} = A_{s''}$ would intersect non-trivially, and thus be the same), and $\alpha_1 \neq \alpha_2$, so $k_1 \neq k_2$ and $j_1 \neq j_2$. However, these are distinct generators, so this cannot happen by Lemma 1.2. Thus this loop is not embedded.

Now suppose we have a loop with 6 edges. This loop has the form

$$\begin{array}{ccccc} & & \beta_1 1 & & \\ & \swarrow & & \searrow & \\ & \alpha_1 A_{\{s_1\}} & & \alpha_3 A_{\{s_3\}} & \\ \nearrow & & & & \nwarrow \\ \beta_2 1 & \longrightarrow & \alpha_2 A_{\{s_2\}} & \longleftarrow & \beta_3 1 \end{array}$$

Since the loop is embedded, each β_i is distinct and at most one of the β_i can be the identity, so assume $\beta_1 \neq 1$ and $\beta_2 \neq 1$. Then since $\beta_1 \neq 1$, we must have that $s_1 \neq s_3$ (as before, if we did have $s_1 = s_3$, then the cosets $\alpha_1 A_{\{s_1\}}$ and $\alpha_3 A_{\{s_3\}}$ would be cosets of the same subgroup $A_{\{s_1\}} = A_{\{s_3\}}$ which intersect non-trivially, thus would be the same coset). Similarly, $s_1 \neq s_2$.

From our diagram, we see that we have equations

$$\alpha_1 s_1^{k_1} = \beta_1 = \alpha_3 s_3^{k_3}$$
$$\alpha_2 s_2^{j_2} = \beta_2 = \alpha_1 s_1^{j_1}$$
$$\alpha_3 s_3^{\ell_3} = \beta_3 = \alpha_2 s_2^{\ell_2}.$$

for some $k_i, j_i, \ell_i \in \mathbb{Z}$. Then we see that

$$\begin{aligned} s_1^{j_1-k_1} &= s_1^{-k_1} s_1^{j_1} \\ &= (\alpha_1^{-1}\beta_1)^{-1}(\alpha_1^{-1}\beta_2) \\ &= \beta_1^{-1}\beta_2, \end{aligned}$$

and similarly,

$$s_2^{\ell_2-j_2} = \beta_2^{-1}\beta_3, \qquad s_3^{k_3-\ell_3} = \beta_3^{-1}\beta_1$$

Note that since the β_i are distinct, none of these exponents are zero. But,

$$\begin{aligned} s_1^{j_1-k_1} &= \beta_1^{-1}\beta_2 \\ &= \beta_1^{-1}(\beta_3\beta_3^{-1})\beta_2 \\ &= (\beta_3^{-1}\beta_1)^{-1}(\beta_2^{-1}\beta_3)^{-1} \\ &= s_3^{\ell_3-k_3} s_2^{j_2-\ell_2}. \end{aligned}$$

This means $s_1^{j_1-k_1} \in A_{\{s_2,s_3\}}$, and so $A_{\{s_1\}} \cap A_{\{s_2,s_3\}} \neq 1$ since $j_1 - k_1 \neq 0$. But then by [vdL83], this would mean $\{s_1\} \cap \{s_2, s_3\} \neq \varnothing$, a contradiction. Thus this loop cannot be embedded.

Therefore, each embedded loop in $\mathrm{lk}_{\hat{\Phi}}(A_i)$ has at least 8 edges. The spherical metric on the link assigns each of these edges a length of $\pi/4$, so the shortest possible length of an embedded loop is 2π.

2.2. **Case 2:** $T = \{s\}$**.** Now we look at the link of $A_{\{s\}}$ with $s \in \mathrm{Vert}(\Gamma_i)$ a vertex of an edge between Γ_i and Γ_j. In this case the link is again a bipartite graph: the vertices can be divided into the sets

$$\begin{aligned} &\{\, \alpha 1 \,:\, \alpha \in A_{\{s\}} \,\}, \qquad \text{and} \\ &\{A_i\} \cup \{\, A_{\{s,s_k\}} \,:\, \{s, s_k\} \text{ is an edge between } \Gamma_i \text{ and } \Gamma_k,\, k \neq i \,\}. \end{aligned}$$

So, every embedded loop has at least 4 edges. The spherical metric on the link assigns a length of $\pi/2$ to each of these edges, implying the length of every embedded loop is at least 2π.

2.3. **Case 3:** $T = \{s_i, s_j\}$**.** The link of A_T for $T = \{s_i, s_j\}$, $s_i \in S_i$, $i \neq j$, is slightly different, as there are two cases to consider. However, in both cases the minimal number of edges in an embedded loop are the same.

Lemma 2.6. *If $T = \{s_i, s_j\}$ is an edge between Γ_i and Γ_j for $i \neq j$, then each embedded loop in $\mathrm{lk}_{\hat{\Phi}}([A_T])$ has at least $4m(s_i, s_j)$ edges.*

Proof. The link of A_T has vertex set which can be split into

$$\{\, \alpha 1 \,:\, \alpha \in A_T \,\} \qquad \text{and} \qquad \{\, \alpha A_{s_k} \,:\, \alpha \in A_T,\, k = i, j \,\},$$

on which the link is a bipartite graph. By applying the natural A_T action on the link, we may consider only loops which contain the vertex 1. Namely, we may consider only loops of the form

$$\begin{array}{ccccccc}
\alpha_1 A_{t_1} & \longleftarrow & \beta_1 & \longrightarrow & \alpha_2 A_{t_2} & \longleftarrow & \beta_2 \\
\big\uparrow & & & & & & \downarrow \\
1 & & & & & & \vdots \\
\downarrow & & & & & & \uparrow \\
\alpha_n A_{t_n} & \longleftarrow & \beta_{n-1} & \longrightarrow & \alpha_n A_{t_{n-1}} & \longleftarrow & \beta_{n-2}
\end{array}$$

where each $\alpha_k \in A_T$ and each $t_k \in T$. This loop has $2n$ edges. Moreover, assuming this loop is embedded, this gives rise to a (reduced) word in s_i and s_j of syllable length at least n (see [AS83, Section 4] for the definition of syllable length) which is equal to the identity in A_T. By [AS83, Lemma 6], this means $n \geq 2m(s_i, s_j)$. Thus, this loop has at least $4m(s_i, s_j)$ edges. □

Now, we can compute the length of these loops in each given link.

2.3.1. *Case 3a: A disjoint edge.* If $\{s_i, s_j\}$ is disjoint from every other edge between the subgraphs in $\{\Gamma_k\}$, then the spherical metric on the link of A_T implies that the length of each edge here is $\pi/4$. So, the length of any embedded loop is at least $(4m(s_1, s_2))(\pi/4) = \pi m(s_1, s_2)$. Since $m(s_1, s_2) \geq 2$, this loop has length at least 2π, as required.

2.3.2. *Case 3b: A non-disjoint edge.* If this edge is not disjoint from every other edge between the subgraphs in $\{\Gamma_k\}$, the metric we have assigned implies that the length of each edge is $\pi/8$. So, the length of any embedded loop is at least $(4m(s_1, s_2))(\pi/8) = \pi m(s_1, s_2)/2$. But in this case, we have also assumed $m(s_1, s_2) \geq 4$, so the length of this loop is still at least 2π.

2.4. **Case 4: $T = \varnothing$.** It remains to check the link of the trivial coset 1. Note again that this link is bipartite, with a partition of the vertices given by

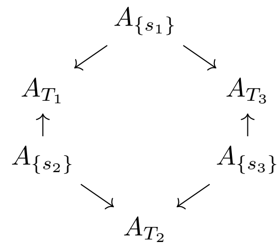
$$\{\, A_{\{s\}} \,:\, s \text{ a vertex of an edge between the subgraphs in } \{\Gamma_k\}\,\}, \qquad \text{and}$$
$$\{\, A_i \,:\, \text{ each } i\,\} \cup \{\, A_{\{s_i, s_j\}} \,:\, \{s_i, s_j\} \text{ an edge between } \Gamma_i \text{ and } \Gamma_j,\, k \neq i\,\}.$$

We first verify that there are no embedded loops with 4 edges. Suppose we had such an embedded loop, say

$$\begin{array}{ccc}
 & A_{\{s\}} & \\
\swarrow & & \searrow \\
A_{T_1} & & A_{T_2} \\
\nwarrow & & \nearrow \\
 & A_{\{s'\}} &
\end{array}$$

Since this loop is embedded, $s \neq s'$. Thus by Lemma 1.1, both T_1 and T_2 contain $\{s, s'\}$. If s and s' are in the same vertex set S_i, then we must have $T_1 = T_2 = S_i$ by our definition of $\mathcal{S}^\ell$. Similarly, if they are in distinct vertex sets, then both T_1 and T_2 must exactly be the edge $\{s, s'\}$. In either case, we have a contradiction.

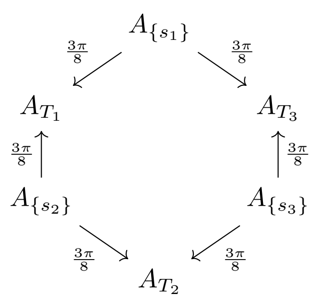
It is entirely possible that we have embedded loops of length 6. Suppose



is such a loop. If each pair $\{s_i, s_j\}$ is an edge between the family of subgraphs $\{\Gamma_i\}$, then the T_i must be the edges

$$T_1 = \{s_1, s_2\},$$
$$T_2 = \{s_2, s_3\},$$
$$T_3 = \{s_3, s_1\}$$

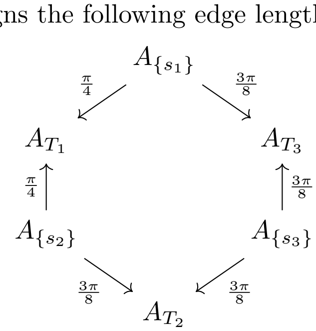
since these are the only sets of $\mathcal{S}^\ell$ which satisfy the containments implied by the diagram. But none of these edges are disjoint, so the metric we've put on $\hat{\Phi}$ assigns the following edge lengths to this path:



And thus this loop has length at least 2π. Now suppose two of the vertices s_i are in the same vertex set and the other is in a distinct vertex set. Without loss of generality, we can take $s_1, s_2 \in S_i$ and $s_3 \in S_j$ with $i \neq j$. Then the only set $T_1 \in \mathcal{S}^\ell$ containing both s_1 and s_2 is $T_1 = S_i$, and so we now have

$$T_1 = S_i,$$
$$T_2 = \{s_2, s_3\},$$
$$T_3 = \{s_3, s_1\}.$$

The metric on $\hat{\Phi}$ then assigns the following edge lengths:



which is still at least 2π. We note that it is not possible to have $s_1, s_2, s_3 \in S_i$ for any i, since then $T_1 = T_2 = T_3 = S_i$, and this loop would not be embedded.

Finally, if we have a loop with 8 edges in this link, then the length of each edge under our metric is at least $\pi/4$, and thus the length of this loop would be at least 2π as well.

This concludes every possibility for T, so it follows that $\hat{\Phi}$ satisfies the link condition by Lemma 2.5. By Lemma 1.4, $\hat{\Phi}$ is simply connected, so by Lemma 2.4, $\hat{\Phi}$ is CAT(0), and thus contractible, as desired.

3. The $K(\pi, 1)$ Conjecture

In this section only, **we assume that each A_{Γ_i} satisfies the $K(\pi, 1)$ conjecture**. In addition, we assume that A_Γ is not spherical-type. (Since the $K(\pi, 1)$ conjecture is known for spherical-type Artin groups, there is no loss of generality in making this assumption.)

We will use the following by Paris and Godelle in [GP12].

Definition 3.1. Let (A, S) be an Artin-Tits system and let $\mathcal{S}$ be a family of subsets of S. Then $\mathcal{S}$ is *complete and* $K(\pi, 1)$ if the following are satisfied:

(1) If $T \in \mathcal{S}$ and $T' \subseteq T$, then $T' \in \mathcal{S}$,
(2) (A_T, T) satisfies the $K(\pi, 1)$ conjecture for each $T \in \mathcal{S}$, and
(3) If A_T is spherical-type, then $T \in \mathcal{S}$.

Then let
$$A\mathcal{S} = \{\, \alpha A_T \,:\, \alpha \in A,\, T \in \mathcal{S} \,\},$$
and let $\Phi(A, \mathcal{S})$ denote the geometric realization of the derived complex of $A\mathcal{S}$.

The relevant result for us is

Theorem 3.2. ***[GP12, Theorem 3.1]*** *Let (A, S) be an Artin-Tits system and let $\mathcal{S}$ be a complete and $K(\pi, 1)$ family of subsets of S. Then $\Phi(A, \mathcal{S})$ has the same homotopy type as the universal cover of $N(W)$.*

Our family $\mathcal{S}^\ell$ is not itself complete and $K(\pi, 1)$, so we cannot directly apply this result. Instead, we show that $\hat{\Phi}$ is homotopic to $\Phi \coloneqq \Phi(A, \overline{\mathcal{S}})$ for a certain complete and $K(\pi, 1)$ collection $\overline{\mathcal{S}}$ which we define as follows: the sets of $\overline{\mathcal{S}}$ are the subsets of S consisting of (1) the sets in $\mathcal{S}^\ell$, and (2) every subset of S_i.

Lemma 3.3. *$\overline{\mathcal{S}}$ is a complete and $K(\pi, 1)$ family of subsets of S.*

Proof. First we note that (1) and (2) are satisfied immediately by our definition of $\overline{\mathcal{S}}$ (to see (2), note that a standard parabolic subgroup satisfies the $K(\pi, 1)$ conjecture whenever the original group does by [GP12, Cor. 2.4]). It remains to show that $\overline{\mathcal{S}}$ contains all spherical-type generating sets.

Suppose Γ' is a full subgraph of Γ such that $A_{\Gamma'}$ is spherical-type, and let $T = \mathrm{Vert}(\Gamma')$. If $T \subseteq S_i$, then we already have $T \in \overline{\mathcal{S}}$. So, suppose there are $t_1, t_2 \in T$ with t_1 and t_2 in distinct vertex sets, say $t_1 \in S_i$ and $t_2 \in S_j$ for $i \neq j$.

If $T = \{t_1, t_2\}$ then since we're assuming $A_{\Gamma'}$ is spherical-type, we must have that $\{t_1, t_2\}$ is an edge of Γ, and thus $T \in \overline{\mathcal{S}}$. In other words, whenever $|T| = 2$ and $T \not\subseteq S_k$ for any k, we must have that T is an edge of Γ, so $T \in \overline{\mathcal{S}}$.

Suppose $|T| > 2$, and let $t_3 \in T$ be distinct from t_1 and t_2. If any of $\{t_1, t_2\}$, $\{t_2, t_3\}$, or $\{t_3, t_1\}$ is not an edge of Γ, then $A_{\Gamma'}$ would not be spherical, so we must have that each of these are edges. There are three cases to consider: either t_3 is in

S_1, is in S_2, or is in neither. By symmetry we may consider only the cases where $t_3 \in S_1$ and t_3 is in neither. In both of these cases, $\{t_1, t_2\}$ and $\{t_3, t_2\}$ are distinct non-disjoint edges between the family $\{\Gamma_i\}$, so by the (REL$'$) condition, both of their labels must be at least 4. By the classification of finite Coxeter groups, we would then have that $A_{\Gamma'}$ is not spherical-type. Thus if $T \not\subset S_i$ we cannot have $|T| > 2$. □

Thus, we have that Φ is homotopy equivalent to the universal cover of $N(W)$. It remains to show that

Theorem 3.4. *There is a deformation retract from Φ to $\hat{\Phi}$.*

Proof. Note that there is a natural embedding of $\hat{\Phi}$ into Φ induced by the inclusion of $\mathcal{S}^\ell$ in $\overline{\mathcal{S}}$.

We establish the deformation retract directly by describing the maps on each simplex. Let Δ be a maximal simplex of Φ (ie, one which is not a face of any other simplex). There are two types of simplices to consider. The first is

$$\Delta = [\alpha_0 1, \alpha_0 A_{\{s\}}, \alpha_1 A_{\{s,t\}}]$$

for an edge $\{s, t\}$ between the family $\{\Gamma_i\}$. This is already a maximal simplex of $\hat{\Phi}$, so we leave it unchanged. In the other case, we have that

$$\Delta = [\alpha_0 A_{T_0}, \alpha_1 A_{T_1}, \ldots, \alpha_{n-1} A_{T_{n-1}}, \alpha_n A_{S_i}]$$

for some S_i. Since Δ is maximal, we must have $T_0 = \varnothing$ and $T_1 = \{s\}$ for some $s \in S_i$. There are two subcases to consider. If s is a vertex of an edge between Γ_i and some Γ_j, then there is a natural deformation retract from Δ to the simplex $[\alpha_0 A_{T_0}, \alpha_1 A_{T_1}, \alpha_n A_{S_i}]$ of $\hat{\Phi}$. Otherwise, there is a natural deformation retract from Δ to the simplex $[\alpha_0 A_{T_0}, \alpha_n A_{S_i}]$ of $\hat{\Phi}$. Moreover, these can easily be parameterized so that we can glue deformation retracts of adjacent maximal simplices to attain a deformation retract on the entire complex Φ. □

We have therefore proven

Theorem A. *Suppose Γ and $\{\Gamma_i\}$ satisfy* (REL$'$)*. Then A_Γ satisfies the $K(\pi, 1)$ conjecture if and only if each A_{Γ_i} does.*

Proof. First, suppose A_Γ satisfies the $K(\pi, 1)$ conjecture. Then, by [GP12, Cor. 2.4], each A_{Γ_i} also does.

Now suppose each A_{Γ_i} satisfies the $K(\pi, 1)$ conjecture. Combining Theorem 3.2 and Lemma 3.3, we have that Φ is homotopy equivalent to the universal cover of $N(W)$, and by Theorem 3.4, $\hat{\Phi}$ is homotopy equivalent to Φ. Thus, by Theorem 2.1, the universal cover of $N(W)$ is contractible. □

4. Acylindrical hyperbolicity

We conclude by showing the following

Theorem B. *Suppose A_Γ and $\{\Gamma_i\}_{i=1}^n$, $n \geq 2$ satisfy* (REL$'$)*. In addition, assume $|\mathrm{Vert}(\Gamma)| \geq 3$ and not all edges between the family $\{\Gamma_i\}$ have label 2. Then A_Γ is acylindrically hyperbolic.*

For the full definition of acylindrical hyperbolicity, we refer the reader to [Bow08].

First, if there are no edges between the family $\{\Gamma_i\}$, then A_Γ is a free product of the A_{Γ_i}, and thus is acylindrically hyperbolic by considering the action of A_Γ on its Bass-Serre tree. So, we assume there is an edge between the family $\{\Gamma_i\}$, say $e = \{s_i, s_j\}$ with $s_i \in S_i$, $s_j \in S_j$ and $i \neq j$. By our assumptions on Γ, we may take e to have label at least 3. In this case, we make use of the following adaptation of a theorem from [Mar17].

Theorem 4.1. **[Mar17, Theorem B]** *Let X be a $\mathrm{CAT}(0)$ simplicial complex and G a group acting on X by simplicial isomorphisms. Suppose there is a vertex v of X with stabilizer G_v satisfying*

(1) *The orbits of G_v on the link $\mathrm{lk}_X(v)$ are unbounded in the associated spherical metric, and*
(2) *G_v is weakly malnormal in G (ie, there exists an element $g \in G$ such that $G_v \cap gG_vg^{-1}$ is finite).*

Then G is either virtually cyclic or acylindrically hyperbolic.

Remark 4.2. This is a strictly weaker statement than the one given in [Mar17]. The original statement of the theorem allows X to be a polyhedral complex satisfying the "Strong Concatenation Property". By Example 2.9 and Lemma 2.11 in [Mar17], CAT(0) simplicial complexes always satisfy this property.

We use the action of A_Γ on our Deligne-like complex $\hat{\Phi}$, which we have shown is CAT(0) for any Artin group satisfying (REL'). We claim that $v_e \coloneqq [A_{\{s_i,s_j\}}]$ is a vertex of $\hat{\Phi}$ which satisfies the conditions of Theorem 4.1.

Since we have assumed $|\mathrm{Vert}(\Gamma)| \geq 3$, there is at least one vertex s of Γ distinct from s_i and s_j. If there are no such s so that either $\{s, s_i\}$ or $\{s, s_j\}$ are edges of Γ, then A_Γ is a free product and thus acylindrically hyperbolic by our previous remarks. In the other case, take s so that one of $\{s, s_i\}$ or $\{s, s_j\}$ is an edge of Γ, and define $\Delta = \{s, s_i, s_j\}$. Then the full subgraph of Γ on vertices Δ is connected, and, by the (REL$'$) condition, A_Δ is a 2-dimensional Artin group. Moreover, we have assumed $m(s_i, s_j) > 2$, so A_Δ is not a right-angled Artin group. Thus, we may use the following

Theorem 4.3. **[Vas21, Lemma 5.7]** *Let A_Λ be a 2-dimensional Artin group of rank at least 3, and suppose that Λ is connected and that A_Λ is not a right angled Artin group. Then there exists an Artin subgroup $A_{\{a,b\}}$ with coefficient $3 \leq m(a,b) < \infty$ and an element $g \in A_\Lambda$ such that $A_{\{a,b\}} \cap gA_{\{a,b\}}g^{-1} = \{1\}$.*

Applying this to A_Δ, we have $a, b \in \Delta$ and $g \in A_\Delta$ so that $A_{\{a,b\}} \cap gA_{\{a,b\}}g^{-1} = \{1\}$. The proof of the Theorem implies that we may take $\{a,b\} = \{s_i, s_j\}$. This shows that v_e satisfies (2). To show v_e satisfies (1), we use

Theorem 4.4. **[Vas21, Lemma 4.5]** *Consider an Artin group $A_{\{a,b\}}$ with coefficient $3 \leq m(a,b) \leq \infty$. Then*

$$\{\, \ell_{\mathcal{S}}(g) \,:\, g \in A_{\{a,b\}} \,\}$$

is unbounded (where $\ell_{\mathcal{S}}(g)$ is the syllable length of g).

By the same analysis in the case of loops, if $\{a, b\}$ is an edge between the family $\{\Gamma_i\}$, then reduced words in a and b correspond to paths in $\mathrm{lk}_{\hat{\Phi}}([A_{\{a,b\}}])$, and vice versa. The edge length of such a path is at least the syllable length of the given word. So, since $m(s_i, s_j) \geq 3$, we have that the action of $A_{\{s_i,s_j\}}$ on $\mathrm{lk}_{\hat{\Phi}}([A_{\{s_i,s_j\}}]) = \mathrm{lk}_{\hat{\Phi}}(v_e)$ is unbounded. Therefore, v_e also satisfies (1), and thus A_Γ is acylindrically hyperbolic.

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