Secondary Bifurcations in Symmetric Systems

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1. INTRODUCTION

Consider an autonomous parameter-dependent system of the form

\[ \dot{x} = f(x, \lambda), \quad (1.1) \]

with \( f : \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}^n \) sufficiently smooth. We will assume that the system is symmetric, that is to say we have

\[ f(\gamma x, \lambda) = \gamma f(x, \lambda) \quad (1.2) \]

for all \( \gamma \) belonging to a compact group \( \Gamma \) of linear operators on \( \mathbb{R}^n \); by standard theory we may assume that \( \Gamma \) is a closed subgroup of the orthogonal group \( O(n) \), and hence also a Lie group. We are mainly interested in the case \( \dim \Gamma > 0 \), since otherwise most of our discussion becomes trivial. We want to study secondary bifurcations for (1.1); by this we mean bifurcations from non-zero equilibria and from non-constant periodic solutions.

When there is no symmetry (i.e. \( \Gamma \) is trivial) the bifurcation problem has been studied using a wide variety of methods which essentially all reduce to a combination of one of the following: the Liapunov-Schmidt method, reduction to center manifolds, Poincaré mappings and normal form theory. Using these methods one obtains for example easily all bifurcations which can appear generically in one-parameter problems (\( k=1 \)); these are saddle-node and Hopf bifurcations at equilibria, and saddle-nodes of limit cycles and period-doublings.
at non-constant periodic solutions (see e.g. Guckenheimer and Holmes (1983)).
In the symmetric case a lot of work has been done about steady-state and Hopf
bifurcations at equilibria which are invariant under the full group $\Gamma$. In
that case the methods mentioned above combine perfectly with a group-theoretic
approach, especially with group representation theory; the outcome has been
a by now well established equivariant bifurcation theory (see e.g. Vanderbauwhede
(1982), Golubitsky, Stewart and Schaeffer (1988)).
The situation changes considerably when one wants to study bifurcations near
equilibria which do not have the full $\Gamma$-symmetry, or near non-constant periodic
solutions. Indeed, such solutions generate, by the group action, a compact
invariant manifold filled with either equilibria or periodic solutions, and
the corresponding "local" bifurcation problem takes a somewhat more global
flavour: one has to study bifurcations near this invariant manifold, and
not near a particular solution on it. As a consequence the classical methods
- Liapunov-Schmidt, center manifold and Poincaré-mapping - are no longer
directly applicable, since in their usual formulation the starting point is
always a particular solution (equilibrium or periodic). So the first step
towards a general bifurcation theory for this case seems to be to establish
appropriate versions of the basic methods, adapted to and incorporating the
symmetries involved in the problem.
In this paper we prove a relatively simple result which clearly indicates
a possible approach. We show that near any group orbit any equivariant vector
field decomposes into two equivariant vector fields; one of these is at each
point tangent to the group orbit through that point, and therefore its flow
is just a "drift" along group orbits; the second component of the decomposition
leaves a normal section to the given group orbit invariant, and its bifurcations
in this normal section generate, via the group action, the bifurcations of the
original vector field. The idea of such a decomposition was first suggested
by Chossat and Golubitsky (1987); the additional frequencies which are a
consequence of the drift along group orbits have already been introduced
In the next section we give a precise formulation of the decomposition result
and discuss its consequences for the bifurcation problem; in section 3 we
prove theorem 1.

2. RESULTS AND DISCUSSION

Let $\Gamma$ be a closed subgroup of $O(n)$ and $L(\Gamma)$ its Lie algebra, i.e. $L(\Gamma)$ is
the tangent space to $\Gamma$ at the identity operator; all elements of $L(\Gamma)$ are
anti-symmetric linear operators on $\mathbb{R}^n$. We also remark that the group $\Gamma$ acts on its Lie algebra by the action
\begin{equation}
(\gamma,\eta) \in \Gamma \times L(\Gamma) \rightarrow \gamma \eta \gamma^{-1} \in L(\Gamma).
\end{equation}

Fix $x_0 \in \mathbb{R}^n$ and let $\Gamma x_0 = \{\gamma x_0 | \gamma \in \Gamma\}$ be the corresponding group orbit. One knows from general theory that $\Gamma x_0$ is a smooth manifold, with tangent space at the point $x_0$ given by $L(\Gamma)x_0 = \{nx_0 | n \in L(\Gamma)\}$. Let $Y$ be the orthogonal complement of $L(\Gamma)x_0$ in $\mathbb{R}^n$; both $L(\Gamma)x_0$ and $Y$ are invariant under the action of the isotropy subgroup of $x_0$, which we denote by $\Sigma_0$:
\begin{equation}
\Sigma_0 := \{\gamma \in \Gamma | \gamma x_0 = x_0\}.
\end{equation}

By the tubular neighborhood theorem (see Bredon (1972) for a general theory, or Vanderbauwhede (1982) for a more direct treatment) there exists a $\Sigma_0$-invariant open neighborhood $\Omega$ of the origin in $Y$ such that:

(i) $U = \{\gamma(x_0+y) | \gamma \in \Gamma, y \in \Omega\}$ is a $\Gamma$-invariant open neighborhood of $\Gamma x_0$ in $\mathbb{R}^n$;

(ii) if $\gamma_1(x_0+y_1) = \gamma_2(x_0+y_2)$ for $\gamma_1, \gamma_2 \in \Gamma$ and $y_1, y_2 \in \Omega$ then $\gamma_1^{-1}\gamma_2 \in \Sigma_0$.

Our main result is then the following (see also Krupa (1988)):

**Theorem 1.** The neighborhood $\Omega$ of the origin in $Y$ can be chosen such that next to the properties (i)-(ii) above also the following holds:
Each $\Gamma$-equivariant vector field $f : U \rightarrow \mathbb{R}^n$ can be written in the form
\begin{equation}
f(\gamma(x_0+y)) = \gamma[\tilde{f}(y) + \tilde{\eta}(y)(x_0+y)]
\end{equation}
where $\tilde{f} : \Omega \rightarrow Y$ and $\tilde{\eta} : \Omega \rightarrow L(\Gamma)$ have the same smoothness properties as $f$, and are $\Sigma_0$-equivariant:
\begin{equation}
\tilde{f}(\sigma y) = \sigma \tilde{f}(y) \quad , \quad \tilde{\eta}(\sigma y) = \sigma \tilde{\eta}(y) \sigma^{-1} \quad , \quad \forall \sigma \in \Sigma_0.
\end{equation}

**Remark 1.** In general the mappings $\tilde{f}$ and $\tilde{\eta}$ will not be uniquely determined by $f$; indeed, if $L(\Sigma_0)$ is the Lie algebra of $\Sigma_0$, and if $\xi : \Omega \rightarrow L(\Sigma_0)$ is any (sufficiently smooth) $\Sigma_0$-equivariant mapping, then (2.3) and (2.4) remain valid if we replace $\tilde{f}$ and $\tilde{\eta}$ by the mappings $\tilde{f}_1 : \Omega \rightarrow Y$ and $\tilde{\eta}_1 : \Omega \rightarrow L(\Gamma)$ defined by
\[ \tilde{f}_1(y) = \tilde{f}(y) - \tilde{z}(y) \cdot (\lambda^{0} + y) \]
\[ \tilde{\eta}_1(y) = \tilde{\eta}(y) + \tilde{z}(y) . \]

Therefore, if \( L(\Lambda_0) \) is nontrivial (i.e., if \( \dim \Lambda_0 > 0 \)) we may have non-uniqueness. Of course, \( \tilde{f}(0) \) is uniquely determined by \( f(\lambda^0) \).

**Remark 2.** When \( f : U \times \Lambda \to \mathbb{R}^n \) is a \( \Gamma \)-equivariant vectorfield depending on a parameter \( \lambda \) in a parameter space \( \Lambda \), then (2.3) and (2.4) hold for each \( \lambda \in \Lambda \), with \( \tilde{f} : \Omega \times \Lambda \to Y \) and \( \tilde{\eta} : \Omega \times \Lambda \to L(\Gamma) \) as smooth as \( f \).

The result of theorem 1 can be formulated in a different way. Because of (2.4) the formula (2.3) says that any \( \Gamma \)-equivariant vectorfield \( f \) can be decomposed into two \( \Gamma \)-equivariant vectorfields \( f_T : U \to \mathbb{R}^n \) and \( f_N : U \to \mathbb{R}^n \), given by

\[ f_T(\gamma(x_0 + y)) := \gamma \tilde{\eta}(\gamma(x_0 + y)) , \quad f_N(\gamma(x_0 + y)) := \gamma \tilde{f}(y) . \quad (2.5) \]

For each \( x = \gamma(x_0 + y) \in U \) we have \( f_T(x) = \tilde{\eta}(x)x \), with \( \tilde{\eta}(x) := \gamma \tilde{\eta}(\gamma^{-1} \in L(\Gamma)) \); this means that \( f_T(x) \) is at each point \( x \in U \) tangent to the group orbit, and hence the flow of \( f_T \) is simply a "drift" along group orbits. The flow of \( f_N \) leaves the normal section \( S := \{ x_0 + y | y \in \Omega \} \) to the group orbit \( \gamma x_0 \) invariant; the flow on \( S \) is described by the reduced \( \Lambda_0 \)-equivariant equation

\[ \dot{y} = \tilde{f}(y) . \quad (2.6) \]

Using (2.3) it is then easily verified (see also Krupa (1988)) that the flow of \( f \) on \( U \) is given by

\[ \tilde{x}(t; \gamma(x_0 + y)) = \gamma \tilde{y}(t; \gamma(x_0 + y)) , \quad (2.7) \]

where \( \tilde{y}(t; y) \) denotes the flow of (2.6), and \( \gamma : \mathbb{R} \times \Omega \to \Gamma \) is the solution of the initial value problem

\[ \dot{y} = \gamma \tilde{\eta}(\gamma(t; y)) , \quad \gamma(0) = \text{Id} . \quad (2.8) \]

It follows that the flow of \( f \) on \( U \) may be understood as the flow of the \( \Lambda_0 \)-equivariant vectorfield \( \tilde{f} \) on \( S \) modulated by "drift along the (group) orbit". Let us now return to the bifurcation problem. Let \( f : \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}^n \) be \( \Gamma \)-equivariant, and suppose that for some parameter-value \( \lambda_0 \) the system (1.1) has
an equilibrium solution \( \dot{x}_0(t) \equiv x_0 \), or, more generally, a solution of "rotating wave" type:

\[
\dot{x}_0(t) = e^{\eta_0 t} x_0 , \quad \forall t \in \mathbb{R} ,
\]

with \( \eta_0 \in \mathbb{L}(\Gamma) \) and \( x_0 \in \mathbb{R}^n \). For \( \lambda = \lambda_0 \) the group orbit \( \Gamma x_0 \) is invariant under the flow, and we want to discuss bifurcation near this invariant manifold.

When we decompose \( f \) near \( \Gamma x_0 \) then we have

\[
\tilde{f}(0,\lambda_0) = 0 \quad \text{and} \quad \tilde{\eta}(0,\lambda_0) = \eta_0 .
\]

i.e. for \( \lambda = \lambda_0 \) the vectorfield \( \tilde{f} \) on \( \Omega \) has an equilibrium \( y = 0 \).

Now suppose that at \( \lambda = \lambda_0 \) an invariant manifold \( M \) bifurcates from \( \Gamma x_0 \); by the equivariance we may assume that \( M \) is \( \Gamma \)-invariant. By (2.7) there exists an \( \tilde{\Gamma} \)-invariant manifold \( M_{\tilde{x}_0} \subset S \) such that \( M \cap S = M_{\tilde{x}_0} \) and \( \tilde{\Sigma}_0(M_{\tilde{x}_0}) = M_{\tilde{x}_0} \). Conversely, each bifurcation from \( y = 0 \) for the reduced vectorfield \( \tilde{f} \) will generate, via the group action, an invariant manifold bifurcating from \( \Gamma x_0 \). So we have reduced the problem to that of the bifurcations from \( y = 0 \) for the vectorfield \( \tilde{f}(y,\lambda) \).

Theorem 1 may be extended by replacing the group orbit \( \Gamma x_0 \) by a \( \Gamma \)-invariant manifold of the form \( \Gamma M \), where \( M \) itself is a compact manifold. The torus of standing waves obtained by Hopf bifurcation in an \( O(2) \)-equivariant system is an example of such a \( \Gamma M \). In this setting we require that a \( \Gamma \)-equivariant vector field on \( M \) decomposes into a \( \Gamma \)-equivariant vector field tangent to the sections \( \gamma M \) and a \( \Gamma \)-equivariant vector field tangent to group orbits. This motivates the following theorem:

**Theorem 2.** Let \( M \subset \mathbb{R}^n \) be a smooth and compact submanifold satisfying the following conditions:

(i) \( T_x M \cap L(\Gamma)x = \{0\} \) for each \( x \in M \);

(ii) all points \( x \in M \) have the same isotropy subgroup \( \Sigma_0 \);

(iii) the sets \( \Gamma_{x,M} := \{ \gamma \in \Gamma | \gamma x \in M \} \) are independent of \( x \in M \), and therefore form a closed subgroup \( \Sigma \) of \( \Gamma \).

Then \( \Gamma(M) := \{ \gamma x | \gamma \in \Gamma , x \in M \} \) is a compact, \( \Gamma \)-invariant submanifold of \( \mathbb{R}^n \).

Moreover, if \( \pi : N \to \Gamma(M) \) is the normal bundle of \( \Gamma(M) \), and \( Y := \pi^{-1}(M) \), then there exists a \( \Sigma \)-invariant open neighborhood \( U \) of \( M \) in \( Y \) such that the following holds:

(a) \( U := \{ \gamma y | \gamma \in \Gamma , y \in \Omega \} \) is an open \( \Gamma \)-invariant neighborhood of \( \Gamma(M) \) in \( \mathbb{R}^n \);

(b) if \( \gamma_1 \gamma_2 = \gamma_2 \gamma_1 \) with \( \gamma_i \in \Gamma \) and \( \gamma_i \in \Omega \) \((i=1,2)\), then \( \gamma_1 \gamma_2 \in \Sigma \);

(c) each \( \Gamma \)-equivariant vectorfield \( f : U \to \mathbb{R}^n \) can be written in the form
\[ f(y) = \gamma[\tilde{f}(y) + \tilde{n}(y)y] \]  
\[ (2.11) \]

with \( \tilde{f} : \Omega \rightarrow T\Omega \) a \( \Sigma \)-equivariant vectorfield over \( \Omega \), and \( \tilde{n} : \Omega \rightarrow L(\Gamma) \) a \( \Sigma \)-equivariant mapping, both with the same smoothness as \( f \).

Hypotheses (i)-(iii) imply that, for \( x \in M \), \( T_x^\Sigma M \) is complementary to \( L(\Gamma)x \) in \( T_x M \) and the projection of \( f(x) \) to \( T_x M \) with kernel \( L(\Gamma)x \otimes N_x (\Gamma M) \) defines a \( \Sigma \)-equivariant vector field. In particular \( M = \{ x_0 \} \) corresponds to theorem 1.

If \( \tilde{x}_0(t) \) is a periodic solution of (1.1) for \( \lambda = \lambda_0 \), and \( \tilde{x}_0(t) \) is not a rotating wave, then one can apply theorem 2 with \( M = \{ \tilde{x}_0(t) \mid t \in \mathbb{R} \} \). As a result, the bifurcation problem near \( \Gamma(M) = \{ \gamma \tilde{x}_0(t) \mid \gamma \in \Gamma, t \in \mathbb{R} \} \) reduces to the bifurcation problem near \( M \) for the reduced \( \Sigma \)-equivariant vector field; moreover, \( \tilde{x}_0(t) \) is, for \( \lambda = \lambda_0 \), still a periodic solution of the reduced equation

\[ \dot{y} = \tilde{f}(y, \lambda) \]  
\[ (2.12) \]

What are now the consequences of our theorems for the basic methods of bifurcation theory? First, it is sufficient to find a center manifold \( \tilde{W}_c \) for \( \tilde{f} \) containing \( M = \{ x_0 \} \) or \( \{ \tilde{x}_0(t) \mid t \in \mathbb{R} \} \), depending on the case, since \( \tilde{W}_c = \{ \gamma x \mid \gamma \in \Gamma, x \in \tilde{W}_c \} \) is then a center manifold through \( \Gamma(M) \). Also, one should construct a Poincaré mapping for \( \tilde{x}_0(t) \) as a periodic solution of (2.12) (and not for the original equation (1.1)); see Chossat and Golubitsky (1987) for an example. By the way, the reduction of (1.1) to (2.12) is in a sense already a kind of Poincaré mapping, although the result is not a mapping but a vectorfield.

Finally, one can apply Liapunov-Schmidt methods to (2.12); since such methods concentrate on steady-state or periodic, this implies (via (2.7) and Floquet theory for (2.8)) that one will obtain solutions of the original equation (1.1) of the form

\[ \tilde{x}(t) = e^{\gamma t} y(t) \]  
\[ (2.13) \]

with \( \gamma \in L(\Gamma) \) and \( y(t) \) a periodic function; if one implements this in the situation of theorem 1 and studies steady-state or Hopf bifurcation for (2.12), then one refinds some of the results of Renardy (1982) on bifurcation from rotating waves. When the starting point is a periodic solution \( \tilde{x}_0(t) \) of (2.12), then one may study subharmonic bifurcation for (2.12) using the approach outlined in Vanderbauwhede (1987). For example, a period-doubling for (2.12) will result in an "invariant-manifold-doubling" for (1.1). The advantage of the Liapunov-Schmidt method is that one can work directly with the original equation, without explicitly making the reduction to (2.12). Indeed, the function \( x(t) \) as given by (2.12) will be a solution of (1.1) if and only if
y(t) is a solution of the equation

\[ \dot{y} = f(y, \eta, \lambda) = f(y, \lambda) - \eta y. \tag{2.14} \]

Since one looks for periodic y(t), one can apply the Liapunov-Schmidt method to (2.14), in which \( \eta \in L(\Gamma) \) appears as a supplementary unknown. The problem is still \( \Gamma \)-equivariant, since

\[ F(\gamma y, \gamma \eta^{-1}, \lambda) = \gamma F(y, \eta, \lambda), \quad \forall \gamma \in \Gamma. \tag{2.15} \]

We hope to report elsewhere on the details of this approach.

3. PROOF OF THEOREM 1

The proof of theorem 1 is based on the following lemma.

**Lemma.** Under the conditions of theorem 1, let \( K \) be a \( \Sigma_0 \)-invariant complement of \( L(\Sigma_0) \) in \( L(\Gamma) \). Then there exists a \( \Sigma_0 \)-invariant neighborhood \( \Omega \) of the origin in \( Y \) and a unique smooth mapping \( \eta^\lambda : \Omega \times \mathbb{R}^n \rightarrow K \) such that

\[ z - \eta^\lambda(y, z)(x_0 + y) \in Y, \quad \forall (y, z) \in \Omega \times \mathbb{R}^n. \tag{3.1} \]

Moreover, \( \eta^\lambda(y, z) \) is linear in its second argument, and also \( \Sigma_0 \)-equivariant:

\[ \eta^\lambda(\sigma y, \sigma z) = \sigma \eta^\lambda(y, z) \sigma^{-1}, \quad \forall \sigma \in \Sigma_0. \tag{3.2} \]

**Proof.** Let \( P \) be the orthogonal projection in \( \mathbb{R}^n \) onto \( L(\Gamma)x_0 = Kx_0 \); then \( Y = \ker P \). Define \( \phi : Y \times \mathbb{R}^n \times K \rightarrow K \) by

\[ \phi(y, z, \eta) := P(z - \eta(x_0 + y)). \tag{3.3} \]

This mapping is smooth, linear in \( (z, \eta) \in \mathbb{R}^n \times K \), with \( \phi(0, 0, 0) = 0 \) and \( D_{\eta}\phi(0, 0, 0) \zeta = -Cx_0 \), such that \( D_{\eta}\phi(0, 0, 0) \) is an isomorphism between \( K \) and \( Kx_0 \). The result then follows from the implicit function theorem and the fact that \( \phi(\sigma y, \sigma z, \sigma \eta^{-1}) = \sigma \phi(y, z, \eta) \) for \( \sigma \in \Sigma_0 \).

Using this lemma the proof of theorem 1 is almost immediate: one simply takes
\[ \tilde{\eta}(y) := \eta^*(y, f(x_0 + y)) \]  
(3.4)

and

\[ \tilde{f}(y) := f(x_0 + y) - \tilde{\eta}(y)(x_0 + y). \]  
(3.5)

The proof of theorem 2 uses essentially the same idea but requires some more technicalities.

REFERENCES


