COINCIDENCE OF HOMEOSTASIS AND BIFURCATION IN FEEDFORWARD NETWORKS

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Homeostasis is an important and common biological phenomenon wherein an output variable does not change very much as an input parameter is varied over an interval. It can be studied by restricting attention to homeostasis points – points where the output variable has a vanishing derivative with respect to the input parameter. In a feedforward network, if a node has a homeostasis point then downstream nodes will inherit it. This is the case except when the downstream node has a bifurcation point coinciding with the homeostasis point. We apply singularity theory to study the behavior of the downstream node near these homeostasis-bifurcation points. The unfoldings of low codimension homeostasis-bifurcation points are found. In the case of steady-state bifurcation, the behavior includes multiple homeostatic plateaus separated by hysteretic switches. In the case of Hopf bifurcation, the downstream node may have limit cycles with a wide range of near-constant amplitudes and periods. Homeostasis-bifurcation is therefore a mechanism by which binary, switch like responses or stable clock rhythms could arise in biological systems.

Keywords: homeostasis, bifurcation, biochemical networks, singularity theory
1. Introduction

Homeostasis and bifurcation are common behaviors in dynamical systems that depend on a parameter. Homeostasis occurs when the output of a system is approximately constant on an interval of the (input) parameter. Bifurcation occurs when the number or type of solutions change as a (bifurcation) parameter is varied. Local bifurcation occurs at an equilibrium when the Jacobian of the system has an eigenvalue with zero real part as the bifurcation parameter varies. Analogously it is shown in [Golubitsky & Stewart, 2016] that homeostasis can be studied by restricting attention to infinitesimal homeostasis points – single points where a component of the system has a vanishing derivative with respect to a parameter – rather than homeostatic intervals. Under this formulation, homeostasis is treated as a singularity of the system. This theory has been used to find homeostatic plateaus in gene regulatory networks [Antoneli et al., 2018] and to explain a paradoxical response in asthma treatment [Donovan, 2018].

Homeostasis singularities

Consider the differential equation \( \dot{X} = G(X, \lambda) \) where \( X \in \mathbb{R}^m \) and \( \lambda \in \mathbb{R} \) is a parameter. We assume there is a stable equilibrium at \( (X, \lambda) = (0, 0) \). We may then apply the implicit function theorem to \( G(X, \lambda) \equiv 0 \) to obtain a curve of stable equilibria as a function of \( \lambda \), say \( X(\lambda) \) where \( X(0) = 0 \). In applications, we are concerned with the homeostatic properties of a distinguished variable \( x \) that we define to be \( X_i \) for some \( i \). We call \( x(\lambda) \) the input-output function and note that it has an infinitesimal homeostasis point at \( \lambda = 0 \) if \( x'(0) = 0 \). Consecutive higher order derivatives may also vanish at the origin, increasing the codimension of the singularity of \( x(\lambda) \). We will use \( \text{codim}_H(x) \) to denote the codimension of \( x(\lambda) \) as a homeostasis point. As codimension increases, we expect the input-output function to be approximately constant on a wider interval and therefore more homeostatic, but low codimension phenomena are more common.

Singularity theory studies the structure of singularities and their perturbations up to appropriate changes of coordinates. It is shown in [Golubitsky & Stewart, 2016] that the appropriate changes of coordinates for homeostasis are those of elementary catastrophe theory. The normal forms for the singularities and their universal unfoldings are reviewed in that paper. The codimension of a singularity is the number of parameters that are found in a universal unfolding of that singularity. The universal unfoldings of the codimension 0 and 1 homeostasis points are listed in table 1.

<table>
<thead>
<tr>
<th>Input-output function, ( x(\lambda) )</th>
<th>Nomenclature</th>
<th>( \text{codim}_H(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \varepsilon \lambda^2 )</td>
<td>Simple Homeostasis</td>
<td>0</td>
</tr>
<tr>
<td>( \varepsilon \lambda^3 + a_1 \lambda )</td>
<td>Chair</td>
<td>1</td>
</tr>
</tbody>
</table>

where \( \varepsilon = \pm 1 \). The normal form for the chair may be recovered by setting \( a_1 = 0 \). The universal unfolding of simple homeostasis is itself.

Bifurcation singularities

We consider bifurcation points with a distinguished bifurcation parameter. Consider a system of the form \( \dot{Y} = F(Y, \mu) \), where \( Y \in \mathbb{R}^n \) and \( \mu \in \mathbb{R} \) is the bifurcation parameter. We assume that \( F \) undergoes a simple 0 eigenvalue steady-state bifurcation or a simple \( \pm i \omega \) eigenvalue Hopf bifurcation at the origin. It is shown in [Golubitsky & Schaeffer, 1985] that either case leads to a scalar equation \( f(y, \mu) = 0 \), \( y \in \mathbb{R} \), whose solutions are in one-to-one correspondence with equilibria of the full system in the case of steady state bifurcation or to amplitudes of limit cycles in the case of Hopf bifurcation. We denote the codimension of a bifurcation point by \( \text{codim}_B(f) \). The universal unfoldings of the codimension 0 and 1 bifurcation points are listed in table 1.
Table 2. Universal unfoldings of low codimension bifurcation points.

<table>
<thead>
<tr>
<th>Bifurcation problem, ( f(y, \mu) )</th>
<th>Nomenclature</th>
<th>codim(_B(f))</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \varepsilon y^2 + \delta \mu )</td>
<td>Limit point</td>
<td>0</td>
</tr>
<tr>
<td>( \varepsilon (y^2 - \mu^2 + b) )</td>
<td>Simple bifurcation</td>
<td>1</td>
</tr>
<tr>
<td>( \varepsilon (y^2 + \mu^2 + b) )</td>
<td>Isola</td>
<td>1</td>
</tr>
<tr>
<td>( \varepsilon y^3 + \delta \mu + b y )</td>
<td>Hysteresis</td>
<td>1</td>
</tr>
</tbody>
</table>

Hopf bifurcations

| \( \varepsilon y^3 + \delta y \mu \) | Simple Hopf | 0 |
| \( \varepsilon (y^3 + y \mu^2 + b y) \) | Isola Hopf  | 1 |
| \( \varepsilon (y^3 - y \mu^2 + b y) \) |              |   |
| \( \varepsilon y^5 + \delta y \mu + b y^3 \) |              | 1 |

where \( \varepsilon = \pm 1 \) and \( \delta = \pm 1 \). The normal forms of codimension 1 bifurcations can be recovered by setting \( b_1 = 0 \). The universal unfolding of a codimension 0 bifurcation is itself.

**Combined homeostasis and bifurcation singularities**

Given the ubiquity of homeostasis and bifurcation, it is natural to study the behavior of systems where the two singularities coexist. However, homeostasis points require the existence of an input-output function, which is not well defined at bifurcation points. We resolve this problem by assuming a feedforward network structure in which the input-output function of a homeostatic system is substituted for the bifurcation parameter in a bifurcating system.

The first equation in the system has the form

\[
\dot{X} = G(X, \lambda) \tag{1}
\]

where \( X \in \mathbb{R}^m \) and \( x \equiv X_i \in \mathbb{R} \) is the distinguished, homeostatic variable of (1) as a function of \( \lambda \in \mathbb{R} \).

The second equation in the system has the form

\[
\dot{Y} = F(Y, x) \tag{2}
\]

where \( Y \in \mathbb{R}^n \). Next suppose \( x(\lambda) \) has a homeostasis point at \( \lambda_0 \) (that is, \( x'(\lambda_0) = 0 \)) and \( x(\lambda_0) = x_0 \). Suppose also that \( F(Y_0, x_0) = 0 \), but \( F \) does not have a steady-state bifurcation at \((Y_0, x_0)\). Then we may apply the implicit function theorem to obtain a curve of equilibria \( Y(x(\lambda)) \in \mathbb{R}^n \) for (2) when \( \lambda \) near \( \lambda_0 \).

Differentiating at \( \lambda_0 \), we have

\[
\frac{d}{d\lambda} Y(x(\lambda)) \bigg|_{\lambda = \lambda_0} = \frac{dx}{d\lambda}(\lambda_0) \frac{dY}{dx}(x_0) = 0 \tag{3}
\]

so that all \( Y \) variables inherit homeostasis points from \( x \). If \( F \) has a bifurcation point, one could vary parameters so that the inherited homeostasis point and the bifurcation point coincide.

We call points \( \lambda_0 \) where infinitesimal homeostasis points \( x(\lambda_0) \) coincide with bifurcation points in \( F(Y, x) \) at \( x(\lambda_0) \) homeostasis-bifurcation points. We are interested in multiplicity and stability of solutions as well as homeostasis points in a particular \( Y \) variable, \( Y_k \). In the case of steady-state bifurcation, we study homeostasis-bifurcation points by reducing the steady-state equation of (2),

\[
F(Y, x) = 0, \tag{4}
\]

to a scalar equation,

\[
f(y, x) = 0 \tag{5}
\]

that preserves the above properties. This can be done if we assume the non-degeneracy condition \( e_k \notin (\ker(DY F)(y_0, x_0))^\perp \) where \( e_k \) is the unit vector in the \( k \)th direction (see appendix 5 for details). In the case
that (2) undergoes a Hopf bifurcation at \((Y_0, x_0)\), a reduction to (5) is still possible, but the non-degeneracy condition becomes \(e_k \notin \langle \text{span}\{\text{Re}(v), \text{Im}(v)\} \rangle \) where \(v\) is an eigenvector of \((D_{YF})_{(Y_0, x_0)}\) whose eigenvalue is purely imaginary.

We define the codimension of a homeostasis-bifurcation point, \(\text{codim}(x, f)\) or \(\text{codim}(f(y, x(\lambda)))\) as the number of parameters needed in its unfolding. Intuitively,

\[
\text{codim}(x, f) = \text{codim}_H(x) + \text{codim}_B(f) + 1
\]

as we need enough parameters to unfold \(x(\lambda)\) and \(f\) individually and then an additional parameter to bring the two singularities together. This formula will be justified in section 3.

**Homeostasis-bifurcation examples**

Examples of biochemical networks with homeostasis-bifurcation points are given in figures 1 (homeostasis-steady state bifurcation) and 2 (homeostasis-Hopf bifurcation). In each case the input-output function is \(X_3(\lambda)\) and \(X_3\) feeds into the \(Y\) network as its input. These networks are analyzed in detail in section 2.
Coincidence of Homeostasis and Bifurcation in Feedforward Networks.

In the context of homeostasis, these points are interesting because the $Y$ variables near these points tend to have more homeostatic properties than a homeostasis point alone would. In the case of steady state bifurcation, the system may have multiple homeostatic plateaus and a natural way to switch between the plateaus as in figure 3. This type of behavior has been found in glycolysis [Mulukutla et al., 2014] (see figure 4), for example. In the case of Hopf bifurcation, the amplitude and period of the limit cycles often have a wide range of near-constant amplitudes and periods as in figure 5. Circadian rhythms have been shown to maintain a period of about 24 hours despite large changes in gene expression levels or variation in temperature [Dibner et al., 2008; Bass & Takahashi, 2010]. Homeostasis-Hopf bifurcation is then a possible mechanism contributing to this phenomenon.

Fig. 2. **A biochemical network exhibiting a chair-isola Hopf point.** $X_3$ has a chair point in $\lambda$ and acts as input to the $Y$ system, which has an isola-Hopf point.

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Fig. 3. **Examples of multiple homeostatic plateaus in homeostasis-steady-state bifurcation.** These diagrams arise from perturbations of the chair-hysteresis point. All perturbation types can be seen in table 5 item (6). The blue (red) curves indicate stable (unstable) equilibria and homeostasis points are marked. These perturbations are particularly interesting because each has two homeostatic plateaus. In (a), the plateau that $y$ lies in is predominantly determined by the current value of $\lambda$, while in (b) the choice of plateau is determined by the history of $\lambda$. 
Fig. 4. **Bistability in cultured HeLa cells** (reproduced from [Mulukutla et al., 2014]). Cells were initially cultured in high glucose (♦) or low glucose (□) and then resuspended in a medium with the indicated glucose concentration. The data suggests the existence of two homeostasis points on the lower branch: one which is apparent in the figure, and one which we would expect to see if it were extended further. The similar glucose consumption rates of both types of cells in very high and very low glucose environments indicate two switches on the border of the plateaus. The homeostasis points and the bistable behavior is suggestive of the behavior depicted in figure 3(a).

![Graph showing bistability in cultured HeLa cells](image)

(a)

(b)

Fig. 5. **Examples of homeostatic amplitude and period in homeostasis-Hopf bifurcation.** These diagrams arise from perturbations of the chair-isola Hopf point. All perturbation types can be seen in table 6 item (11)⁺. The blue (red) curves indicate stable (unstable) equilibria. Green curves indicate the maxima and minima of stable limit cycles. Homeostasis points are marked.

![Diagrams illustrating homeostatic amplitude and period](image)

(a)

(b)

In the context of bifurcation, homeostasis-bifurcation points are interesting because the feedforward structure restricts the space of allowable perturbations, therefore reducing the codimension. As a result, these points provide a way to access high codimension behavior through a low codimension singularity.
For example, $\text{codim}_B(y^3 + \mu^3) = 5$, but $\text{codim}(\lambda^3, y^3 + \mu) = 3$. This is significant because low codimension phenomena are more common in applications.

**Organization of this paper**

In section 2, we analyze the systems resulting from the networks in figures 1 and 2. In section 3 we apply the unfolding theorems from elementary catastrophe theory and bifurcation theory to characterize the universal unfoldings of homeostasis-bifurcation points. In section 4 we define the transition varieties and use standard singularity theory methods to prove these are the only transitions that can occur. In section 5 we use the theory developed in sections 3 and 4 to find the universal unfoldings and persistent phenomena for the homeostasis-bifurcation points arising from the singularities in tables 1 and 1. The paper ends with appendix 5 describing the needed Lyapunov-Schmidt reduction in the context of combined homeostasis-bifurcation singularities.

**2. Examples of homeostasis-bifurcation**

In this section we discuss two examples of homeostasis-bifurcation arising from the networks in figure 1 and figure 2. In each case we find a homeostasis-bifurcation point and all possible behaviors arising from perturbations away from this point. That these are in fact all possible behaviors is justified in sections 3-6.

**Example 1: chair-hysteresis in network of figure 1**

Consider the network depicted in figure 1. The $X$ network is the feedforward excitation network of [Reed et al., 2017]. We assume mass action kinetics in the $X$ component except for the degradation rate of $X_3$, which is determined by the feedforward function, $\eta$. In the $Y$ network, $Y_1$ catalyzes the reaction $Y_2 \rightarrow \tilde{Y}_2$ and $Y_2$ catalyzes the degradation of $Y_1$. $Y_1$ is also degraded at a basal rate independent of $Y_2$. We assume Michaelis-Menten kinetics in the reactions between $Y_2$ and $\tilde{Y}_2$ but mass action otherwise. Letting $x_i$ and $y_i$ denote the concentration of $X_i$ and $Y_i$, respectively, the differential equations for the network are given by

$$
\begin{align*}
\dot{x}_1 &= \lambda - 2x_1 \\
\dot{x}_2 &= x_1 - 2x_2 \\
\dot{x}_3 &= x_2 - (1 + \eta(x_1))x_3
\end{align*}
$$

$$
\begin{align*}
\dot{y}_1 &= x_3 - 2y_1y_2 - y_1 \\
\dot{y}_2 &= -\frac{y_1y_2}{1 + y_2} + \frac{\tilde{y}_2}{b + \tilde{y}_2} \\
\dot{\tilde{y}}_2 &= \frac{y_1y_2}{1 + y_2} - \frac{\tilde{y}_2}{b + \tilde{y}_2}
\end{align*}
$$

where $\eta(x) = \frac{1}{1+\gamma(x)}$ and $\gamma(x) = e^{\frac{-a-x}{b}}$. $\lambda$ is the input parameter while $a$, $c$, and $b$ are auxiliary parameters. The output variable of the $X$ network is $x_3$. The steady states also depend on the initial condition $y_2(0) + \tilde{y}_2(0)$, which we set to 5.

Repeating the analysis in [Reed et al., 2017] shows that if $a = c/6$ then $x_3(\lambda)$ has a chair point at $\lambda_0 = 2c$ with $x_3(\lambda_0) = c/3$. For the $Y$ network, if we treat $x_3$ as the bifurcation parameter, then we find a hysteresis point when $b = b^* \approx 16.91$ by using the numerical continuation software MatCont [Dhooge et al., 2008]. Fix $b$ at $b^*$, and let $x_3^2$ be the value of $x_3$ at the hysteresis point. The network will have a chair-hysteresis point if $x_3(\lambda_0) = x_3^2$. Choosing $c = 3x_3^2$, therefore produces a chair-hysteresis point. Figure 6 shows $x_3(\lambda)$, the equilibria of $y_1$ as a function of $x_3$, and the equilibria of $y_1$ as a function of $\lambda$ at the chair-hysteresis point.
Fig. 6. **The diagrams of $x_3$ and $y_1$ at the chair-hysteresis point.** (a): The marked point indicates the chair point for $x_3(\lambda)$. (b): The marked point indicates the hysteresis point of the $Y$ network with $x_3$ as the bifurcation parameter. (c): The chair-hysteresis point is marked. Neither homeostasis nor hysteresis is visible because the two singularities annihilate each other when they coincide.

There are nine persistent perturbations of a chair-hysteresis point which are enumerated in Table 5 item (6). By choosing the parameters $a$, $b$, and $c$ in (7) and (8) appropriately, we can reproduce all of these behaviors in the network (figure 7). The behaviors shown in 7(c) and 7(d) (and highlighted in figure 3) are of particular interest because each has two stable homeostatic plateaus corresponding to a low state and a high state. In the parameter region corresponding to figure 7(c), there are three hysteretic switches. The middle switch allows for switching between the two homeostatic plateaus while the outer switches define where the system escapes homeostasis. In the parameter region corresponding to figure 7(d), the low and high plateaus coexist over the same range of $\lambda$. In this case the state of the system would depend on the history of the input rather than its current value. This behavior could be desirable if it takes energy to move $\lambda$ outside of the plateau region and without any external forcing $\lambda$ remains near the center of the plateau. The state of $y_1$ could then be controlled by bumping $\lambda$ in the appropriate direction and then letting it relax back to center.
Coincidence of Homeostasis and Bifurcation in Feedforward Networks.

Fig. 7. Behavior of the chair-hysteresis network of figure 1. Each diagram corresponds to a persistent perturbation in Table 5 item (6). The blue (red) curves indicate stable (unstable) equilibria and homeostasis points are marked. The parameters chosen to construct each diagram are (a): $a = .68, b = 12, c = 4.45$; (b): $a = .68, b = 12, c = 4.5$; (c): $a = .65, b = 12, c = 4.5$; (d): $a = .73, b = 12, c = 4.51$; (e): $a = .67, b = 12, c = 4.52$; (f): $a = .67, b = 12, c = 4.55$; (g): $a = .8, b = 12, c = 4.5$; (h): $a = .6, b = 18, c = 3.3$; (i): $a = .5, b = 18, c = 3.25$.

Example 2: chair-isola Hopf in network of figure 2

Now consider the network depicted in figure 2. The $X$ network is the same as above, and the $Y$ network is adapted from [Duncan et al., 2018]. We assume mass action kinetics for the $Y$ network except for the reaction $Y_1 \rightarrow Y_2$, which is controlled by the feedback function $\zeta$. The differential equations for the $X$
network are given by (7), and the equations for the $Y$ network are

$$
\begin{align*}
\dot{y}_1 &= x_3 - \zeta(y_4)y_1 \\
\dot{y}_2 &= \zeta(y_4)y_1 - y_2 \\
\dot{y}_3 &= y_2 - y_3 \\
\dot{y}_4 &= y_3 - y_4
\end{align*}
$$

(9)

where we take $\zeta(y) = 10/(1 + y^{10}) + b$. Using MatCont [Dhooge et al., 2008], we find that there is an isola Hopf bifurcation when $b = b^* \approx .011$. There are two simple Hopf bifurcations connected by a branch of stable limit cycles when $0 < b < b^*$ and there are no Hopf bifurcations when $b > b^*$.

As before, the input-output function is $x_3(\lambda)$. We take the distinguished $Y$ variable to be $y_4$. Letting $x^{H}_3$ be the value of $x_3$ at the isola Hopf bifurcation, there is a chair-isola Hopf point at $\lambda_0 = 2c$ if $c = 3x^{H}_3$ and $a = c/6$. Figure 8 shows the equilibrium values of $y_4$ at the singularity.

![Diagram of $y_4$ at the chair-isola Hopf point. The singularity is marked.](image)

There are 13 persistent perturbations of the chair-isola Hopf which are enumerated in Table 5 item $(11)^+$. The corresponding diagrams for the network are shown in figure 10. The behaviors shown in figures 10(d), 10(f), 10(h) (and highlighted in figure 5) are particularly interesting from the perspective of homeostasis. In figure 10(f), the limit cycle amplitudes are exceptionally homeostatic with 5 homeostasis points between the two Hopf bifurcations. In each of figures 10(d), 10(f), 10(h) the limit cycle periods are homeostatic with two homeostasis points. Homeostatic period of limit cycles is desirable in biological clocks, for example.
Fig. 9. Behavior of the chair-isola Hopf network in figure 2. Each diagram corresponds to a persistent perturbation in Table 5 item (11). The blue (red) curves indicate stable (unstable) equilibria. The green curves indicate the maxima and minima of the stable limit cycles. The middle and bottom graphs show the amplitude and period of the limit cycles, respectively. Homeostasis points are marked. The parameters chosen to construct each diagram are (a): $a = .7, b = .01, c = 4.6$; (b): $a = .65, b = .01$; (c): $a = .6, b = .01, c = 4.75$; (d): $a = .73, b = .01, c = 4.78$; (e): $a = .73, b = .01, c = 4.78$; (f): $a = .73, b = .01, c = 4.8$. 
Fig. 10. Behavior of the chair-isola Hopf network in figure 2 (continued). The parameters are (g): $a = .65$, $b = .01$, $c = 4.88$; (h): $a = .75$, $b = .01$, $c = 4.85$; (i): $a = .7$, $b = .01$, $c = 4.93$; (j): $a = .78$, $b = .01$, $c = 5$; (k): $a = .9$, $b = .01$, $c = 4.78$; (l): $a = .82$, $b = .11$, $c = 4.78$; (m): $a = .7$, $b = .11$, $c = 4.78$. 
3. Universal Unfoldings

In this section, we characterize the universal unfolding of homeostasis-bifurcation points. The allowable perturbations are those which respect the feedforward structure of (1)-(2). That is, (1) and (2) can be independently perturbed, but the input-output function can never depend on the state variables of (2). For this reason, we can independently unfold the input-output function, \( x \), and the bifurcation problem, \( f \), and then link them together to obtain the unfolding of the homeostasis-bifurcation point, \((x, f)\).

First, we unfold the homeostasis point. Let \( x(\lambda, a) \) be a family of input-output functions where \( \lambda \) is the input parameter and \( a \) parameterizes the family. \( a \neq 0 \) represents a perturbation away from the homeostasis point. Specifically, suppose \( x \) has a \( \lambda \)-homeostasis point at \((\lambda, a) = (0, 0)\) with \( x(0, 0) = 0\). By the universal unfolding theorem in elementary catastrophe theory, \( x(\lambda, a) \) factors through \( \pm \lambda^k + a_{k-2}\lambda^{k-2} + \cdots + a_1\lambda \) where \( k \) is the first non-vanishing \( \lambda \)-derivative of \( x \) and \( \text{codim}_H(x) = k - 2 \) (see [Golubitsky & Stewart, 2016]). In particular we have

\[
x(\lambda, a) = \pm \Lambda(\lambda, a)^k + A_{k-2}(a)\Lambda(\lambda, a)^{k-2} + \cdots + A_1(a)\Lambda(\lambda, a) - C(a)
\]  

where \( \Lambda(0, 0) = 0, A(0) = 0, C(0) = 0, \) and \( \Lambda_\lambda > 0 \).

Next, we unfold the bifurcation point. Let \( f(y, \mu, b) \) be a family of functions with a bifurcation point at \((y, \mu, b) = (0, 0, 0)\). As before, \( b \) parameterizes the family and \( b \neq 0 \) indicates a perturbation from the bifurcation. Define \( \ell := \text{codim}_B(f) \). By the universal unfolding theorem for bifurcations, \( f(y, \mu, b) \) factors through a normal form, say \( N(y, \mu, B) \) where \( B \in \mathbb{R}^\ell \). That is,

\[
f(y, \mu, b) = S(y, \mu, b)N(Y(y, \mu, b), M(\mu, b), B(b))
\]

where \( S(0, 0, 0) = 0, Y(0, 0, 0) = 0, M(0, 0) = 0, B(0) = 0, Y_g(0, 0, 0) > 0, \) and \( M_g(0, 0) > 0 \) (see [Golubitsky & Schaeffer, 1985]).

A homeostasis-bifurcation point is created by linking the input-output function, \( x(\lambda) \), with the bifurcation problem, \( f(y, \mu) \), to form \( h(y, \lambda) := f(y, x(\lambda)) \). Replacing \( x \) and \( f \) with their normal form shows that \( h \) factors as

\[
h(y, \lambda, a, b, c) = S(x(\lambda, a), y, b)N(Y(x(\lambda, a), y, b), X(x(\lambda, a), b), B(b)).
\]

Therefore a universal unfolding for \( h \) is given by

\[
N(y, \pm \lambda^k + a_{k-2}\lambda^{k-2} + \cdots + a_1\lambda - c, b).
\]

Noting that there are \((k - 2) + \ell + 1 = \text{codim}_H(x) + \text{codim}_B(f) + 1 \) parameters in this unfolding justifies formula (6).

4. Transition varieties

In this section we define the transition varieties of a homeostasis-bifurcation point. These are the set of points in parameter space across which the diagram qualitatively changes. Knowledge of these transition varieties allows us to find all possible behaviors near the homeostasis-bifurcation point.

Consider a perturbed version of (1)-(2):

\[
\dot{X} = G(X, \lambda, a)
\]

\[
\dot{Y} = F(Y, x - c, b)
\]

where we recall \( X \in \mathbb{R}^m, Y \in \mathbb{R}^n, \) and \( x = X_1 \) is the distinguished, homeostatic variable of (14) and \( Y_k \) is the variable of interest. We assume (15) undergoes a steady-state or Hopf bifurcation at \((Y, x, b, c) = (0, 0, 0, 0)\). Let \( x(\lambda, a) \) be the input output function of (14) and \( f(y, \mu, b) \) be an appropriate scalar reduction of (15) at the bifurcation point. Define \( h(y, \lambda, a, b, c) = f(y, x(\lambda, a) - c, b) \). Let \( U, L \subset \mathbb{R} \) be closed intervals and \((a, b, c) \in W \subset \mathbb{R}^k \). We take \( U \times L \times W \) to be the domain of \( h \). Solutions to \( f = 0 \) correspond to equilibria or amplitudes of limit cycles.

We will need the following definition from [Golubitsky & Schaeffer, 1985].

Definition 4.1. A branch of \( h \) is a continuous function \( C : [L_1, L_2] \to U \) which is smooth on \((L_1, L_2)\) and satisfies
(1) \( h(C(\lambda), \lambda) = 0 \) on \([L_1, L_2]\), and
(2) either \( L_i \in \partial L \) or \((C(L_i), L_i)\) is a bifurcation point of \( h \) for \( i = 1, 2 \).

When (15) has a Hopf bifurcation, phase-shift symmetry forces \( f \) to be odd in \( y \) and that symmetry must be respected by perturbations. As a result, the transition varieties will differ depending on whether (15) has a steady-state or Hopf bifurcation. We first define the transition varieties for a steady-state bifurcation.

The transition varieties for homeostasis-steady state bifurcation can be defined by conditions on \( h \) or by conditions on \((x, f)\). Defined by conditions on \( h \) they are as follows.

\[
\mathcal{B} = \{(a, b, c)|\exists (y, \lambda) \text{ such that } h = h_\lambda = h_y = 0\}
\]

\[
\mathcal{H} = \{(a, b, c)|\exists (y, \lambda) \text{ such that } h = h_y = h_{yy} = 0\}
\]

\[
\mathcal{D} = \{(a, b, c)|\exists (y_1, y_2, \lambda) \text{ such that } h = h_y = 0 \text{ at } (y_i, \lambda), i = 1, 2\}
\]

\[
\mathcal{C} = \{(a, b, c)|\exists (y, \lambda) \text{ such that } h = h_\lambda = h_{\lambda\lambda} = 0\}.
\]

Note that \( \mathcal{B}, \mathcal{H}, \) and \( \mathcal{D} \) are the bifurcation, hysteresis, and double limit point transition varieties for \( h \) as a bifurcation point (see chapter 3 of [Golubitsky & Schaeffer, 1985]). \( \mathcal{C} \) is the parameter set where a branch of \( h \) has a chair point.

In terms of \( x \) and \( f \) the transition varieties are

\[
CH_x = \{(a, b, c)|\exists (x, y, \lambda) \text{ such that } \mu - x = f = x_\lambda = x_{\lambda\lambda} = 0\}
\]

\[
\mathcal{B} = \{(a, b, c)|\exists (x, y, \lambda) \text{ such that } \mu - x = f = f_\mu = f_y = 0\}
\]

\[
HYS = \{(a, b, c)|\exists (x, y, \lambda) \text{ such that } \mu - x = f = f_y = f_{yy} = 0\}
\]

\[
\mathcal{D} = \{(a, b, c)|\exists (x, \lambda, y_1, y_2), y_1 \neq y_2 \text{ such that } \mu - x = f = f_y = 0 \text{ at } (x, y_i, \lambda), i = 1, 2\}
\]

\[
CH_f = \{(a, b, c)|\exists (x, y, \lambda) \text{ such that } \mu - x = f = f_\mu = f_{\mu\mu} = 0\}
\]

\[
\mathcal{C} = \{(a, b, c)|\exists (x, y, \lambda) \text{ such that } \mu - x = f = f_y = x_\lambda = 0\}
\]

\[
HH = \{(a, b, c)|\exists (x, y, \lambda) \text{ such that } \mu - x = f = x_\lambda = f_\mu = 0\}.
\]

\( CH_x \) is the set of parameters where \( x \) has a chair point. \( \mathcal{B}, HYS, \) and \( \mathcal{D} \) are the transition varieties of \( f \) as a bifurcation point. \( CH_f \) is where \( f \) has a branch with a chair point. \( \mathcal{C} \) and \( HH \) are due to interactions between \( x \) and \( f \). \( \mathcal{C} \) is the coincidence transition variety consisting of points where homeostasis in \( x \) coincides with bifurcation in \( f \). \( HH \) is where homeostasis in \( x \) coincides with homeostasis on a branch of \( f \).

**Proposition 1.** The two above definitions for the transition varieties are equivalent. That is, \( CH_1 \cup \mathcal{B} \cup HYS \cup \mathcal{D} \cup \mathcal{C} \cup HH \cup CH_2 = \mathcal{B} \cup \mathcal{H} \cup \mathcal{D} \cup \mathcal{C} \).

**Proof.** It is clear that \( \mu - x(\lambda, a) = f(y, \mu - c, b) = 0 \) if and only if \( h(y, \lambda, a, b, c) = f(y, x(\lambda, a) - c, b) = 0 \).

Next, notice \( h_\lambda = f_\mu x_\lambda, h_y = f_y, h_{yy} = f_{yy}, \) and \( h_{\lambda\lambda} = f_{\mu\mu} x_\lambda^2 + f_\mu x_{\lambda\lambda} \) so that we have

1. \( h_\lambda = h_y = 0 \) if and only if \((f_x = 0 \text{ or } x_\lambda = 0) \) and \( f_y = 0, \)
2. \( h_y = h_{yy} = 0 \) if and only if \( f_y = f_{yy} = 0, \)
3. \( h_y = 0 \) if and only if \( f_y = 0, \)
4. \( h_\lambda = h_{\lambda\lambda} = 0 \) if and only if \((x_\lambda = f_\mu = 0) \) or \((f_\mu = f_{\mu\mu} = 0) \) or \((x_\lambda = x_{\lambda\lambda} = 0). \)

Each of these, respectively, imply \( \mathcal{B} = \mathcal{B} \cup \mathcal{C}, \mathcal{H} = HYS, \mathcal{D} = \mathcal{D}, \) and \( \mathcal{C} = HH \cup CH_2 \cup CH_1. \)

In addition to these transition varieties, there are transitions which correspond to singularities occurring on the boundary of \( U \times L. \) For bifurcations, these are described in chapter 4 of [Golubitsky & Schaeffer,
for some function \(Y\) obtain a curve of equilibria, as appropriate. To simplify notation, collect the parameters into a single variable \(CH\) accounts for chair points of the amplitude, and \(Y\) equilibrium homeostasis points of \(F\) so we will assume that these boundary transitions do not occur.

Now suppose (15) undergoes a Hopf bifurcation at \((Y, x, b, c) = (0, 0, 0, 0)\). In this case we make an additional non-degeneracy assumption. We may apply the implicit function theorem to \(F(Y, x, 0) = 0\) to obtain a curve of equilibria, \(Y(x)\). We assume \(Y(x) \neq 0\). This assumption guarantees that locally the only equilibrium homeostasis points of \(Y\) are inherited by \(x\). The reduction of (15) yields \(f(y, \mu, b) = \rho(y^2, \mu, b)y\) for some function \(\rho(u, \mu, b)\). Define \(r(u, \lambda, a, b, c) = \rho(u, x(\lambda, a) - c, b). h\) is then given by \(h(y, \lambda, a, b, c) = r(y^2, \lambda, a, b, c)y\). The transition varieties can be defined by conditions on \(r\) and \(x\) only. They are as follows.

\[
\begin{align*}
B_H &= \{(a, b, c) \mid \exists (u, \lambda), u > 0 \text{ such that } r = r_\lambda = r_u = 0 \} \\
B_0 &= \{(a, b, c) \mid \exists \lambda \text{ such that at } u = 0, r = r_\lambda = 0 \} \\
\mathcal{H}_H &= \{(a, b, c) \mid \exists (u, \lambda), u > 0 \text{ such that } r = r_u = r_{uu} = 0 \} \\
\mathcal{H}_0 &= \{(a, b, c) \mid \exists \lambda \text{ such that at } u = 0, r = r_u = 0 \} \\
\mathcal{D}_H &= \{(a, b, c) \mid \exists (u_1, u_2, \lambda)u \geq 0 \text{ such that } r = r_u = 0 \text{ at } (u_i, \lambda), i = 1, 2 \} \\
\mathcal{C}_H &= \{(a, b, c) \mid \exists (u, \lambda), u > 0 \text{ such that } r = r_\lambda = r_{\lambda\lambda} = 0 \} \\
CH_H &= \{(a, b, c) \mid \exists \lambda \text{ such that } x_\lambda = x_{\lambda\lambda} = 0 \}.
\end{align*}
\]

These are the standard transition varieties for \(h\) treated as a bifurcation with the addition of \(\mathcal{C}_H\), which accounts for chair points of the amplitude, and \(CH_H\) which accounts for chair points on the equilibria.

We use the definition involving \(h\) for the steady-state transitions in what follows. Define

\[\Sigma = \mathcal{B} \cup \mathcal{H} \cup \mathcal{D} \cup \mathcal{C} \quad \text{or} \quad \Sigma = \mathcal{B}_H \cup \mathcal{B}_0 \cup \mathcal{H}_H \cup \mathcal{H}_0 \cup \mathcal{D}_H \cup \mathcal{C}_H \cup CH_H\]

as appropriate. To simplify notation, collect the parameters into a single variable \(\alpha = (a, b, c) \in W\). We will show that if \((15)\) \(\alpha, \beta \in W \setminus \Sigma\) are in the same connected component of \(W \setminus \Sigma\), then \(h(y, \lambda, \alpha)\) and \(h(y, \lambda, \beta)\) have the same diagram. Because \(\Sigma\) is a superset of the bifurcation transition varieties we immediately have

**Proposition 2.** Let \(\alpha\) and \(\beta\) be in the same connected component of \(W \setminus \Sigma\) and suppose there are no boundary transitions. Then \(h(\cdot, \cdot, \alpha)\) and \(h(\cdot, \cdot, \beta)\) are equivalent as bifurcation problems.

**Proof.** See Theorem 10.1 in chapter 3 and Theorem 4.1 in chapter 6 of [Golubitsky & Schaeffer, 1985].

Equivalence as bifurcations means \(h(\cdot, \cdot, \alpha)\) and \(h(\cdot, \cdot, \beta)\) have the same bifurcation diagram. In particular, it allows us to identify branches of \(h(\cdot, \cdot, \alpha)\) and \(h(\cdot, \cdot, \beta)\) with each other. This identification allows us to determine if the branches have the same number of homeostasis points.

**Theorem 1.** Let \(\alpha\) and \(\beta\) be in the same connected component of \(W \setminus \Sigma\) and suppose there are no boundary transitions. Then if \(C^\alpha\) and \(C^\beta\) are corresponding branches of \(h(\cdot, \cdot, \alpha)\) and \(h(\cdot, \cdot, \beta)\) with homeostasis points \(\nu_1 < \nu_2 < \cdots < \nu_i\) and \(\sigma_1 < \sigma_2 < \cdots < \sigma_j\) respectively, then \(i = j\) and \(\text{sign}(C^\alpha''(\nu_m)) = \text{sign}(C^\beta''(\sigma_m))\) for each \(m\).

**Proof.**

Differentiating \(h(C^\alpha(\lambda), \lambda, \alpha)\) in \(\lambda\) and using \(\alpha \notin \mathcal{C}\) or \(\alpha \notin \mathcal{C}_H\) shows that \(C^\alpha'\) and \(C^\alpha''\) cannot simultaneously vanish. The same statement is true for \(C^\beta\). This fact and the compactness of the domains of \(C^\alpha\) and \(C^\beta\) together implies that there are only a finite number of homeostasis points on each branch. Therefore the enumeration in the statement of the theorem is well defined.

Let \(\alpha(t)\) be a path in a connected component of \(W \setminus \Sigma\) with \(\alpha(0) = \alpha\) and \(\alpha(1) = \beta\). For each \(t \in [0, 1]\), we can identify a branch of \(h(\cdot, \cdot, \alpha(t))\) with \(C^\alpha\) because the reduced functions are equivalent as bifurcations by Proposition 2. Name this branch \(C^t\). Note that \(C^0 = C^\alpha\) and \(C^1 = C^\beta\).
Let $t_0 \in [0, 1]$ and $\lambda_1 < \lambda_2 < \cdots < \lambda_{\ell}$ be the set of points where $\phi(\lambda, t_0) := C^{t_0}(\lambda)$ vanishes. For each $i$, $\phi(\lambda_i, t_0) = 0$ and $\phi(\lambda_{i+1}, t_0) = C^{t_0}''(\lambda_i) \neq 0$. So, by the implicit function theorem, there is a smooth curve, $\Lambda_i(t)$ so that $\phi(\Lambda_i(t), t) = 0$ for $t$ near $t_0$. We can construct such a function for any $t_0 \in [0, 1]$. By the compactness of $[0, 1]$ we can patch together these curves and define $\Lambda_i(t)$ globally on $[0, 1]$. Uniqueness, implied by the implicit function theorem, then guarantees that $\ell = i = j$ and the ordering is preserved as the curves can’t cross.

For each $t$, $C^t(\Lambda_i(t)) = 0$ and $C^t''(\Lambda_i(t)) \neq 0$. So by continuity, $\text{sign}(C^t''(\Lambda_i(t)))$ is constant and in particular $\text{sign}(C^t''(\nu_i)) = \text{sign}(C^t''(\sigma_i))$.

Proposition 3 and Theorem 2 together imply that the diagrams of $h$ are qualitatively the same on connected components of $W \setminus \Sigma$ in the case of steady-state bifurcation. For Hopf bifurcation, this does not rule out homeostasis transitions for the equilibria solutions. Our assumption that the equilibria $Y^\prime_k(x) \neq 0$ at the Hopf bifurcation means homeostasis in $Y_k(x(\lambda))$ can only be inherited from $x$. $CH_H$ thus accounts for these transitions. It is possible that (14)-(15) have the same diagram on different connected components. Indeed section 5 contains examples of this.

5. Low codimension homeostasis-bifurcation points

In this section we provide all information for the homeostasis-bifurcation points arising from the singularities in Tables 1 and 1. The information is organized into Tables 3-6 with the numbers in parentheses indicating how to link the information between tables. When plotting the diagrams of persistent phenomena, we do not plot the period of limit cycle solutions. However, homeostasis points of the period which are inherited by $x$ can be recovered by noting that these coincide with homeostasis points on the branch of unstable equilibria (see figure 5 for an example). We assume that $f_y < 0$ indicates a stable equilibrium.
Table 3. Defining conditions

<table>
<thead>
<tr>
<th>Normal Form</th>
<th>Defining Conditions*</th>
<th>Nondegeneracy Conditions</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>Homeostasis-steady state bifurcations</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(1) ( x(\lambda) = \eta \lambda^2 )</td>
<td>( f(y, \mu) = \varepsilon y^2 + \delta \mu )</td>
<td>( \eta = \text{sign}(x_{\lambda\lambda}) ) ( \varepsilon = \text{sign}(f_y), \delta = \text{sign}(f_{\mu}) )</td>
</tr>
<tr>
<td>(2) ( x(\lambda) = \eta \lambda^2 )</td>
<td>( f(y, \mu) = \varepsilon(y^2 + \delta \mu^2) )</td>
<td>( f_\mu = 0 ) ( \eta = \text{sign}(x_{\lambda\lambda}) ) ( \varepsilon = \text{sign}(f_y), \delta = \text{sign}(f_{\mu}) )</td>
</tr>
<tr>
<td>(3) ( x(\lambda) = \eta \lambda^2 )</td>
<td>( f(y, \mu) = \varepsilon y^3 + \delta \mu )</td>
<td>( f_{yy} = 0 ) ( x_{\lambda\lambda} = 0 ) ( \eta = \text{sign}(x_{\lambda\lambda\lambda}) ) ( \varepsilon = \text{sign}(f_y), \delta = \text{sign}(f_{\mu}) )</td>
</tr>
<tr>
<td>(4) ( x(\lambda) = \eta \lambda^3 )</td>
<td>( f(y, \mu) = \varepsilon y^3 + \delta \mu )</td>
<td>( x_{\lambda\lambda} = 0 ) ( f_\mu = 0 ) ( \eta = \text{sign}(x_{\lambda\lambda\lambda}) ) ( \varepsilon = \text{sign}(f_y), \delta = \text{sign}(f_{\mu}) )</td>
</tr>
<tr>
<td>(5) ( x(\lambda) = \eta \lambda^3 )</td>
<td>( f(y, \mu) = \varepsilon y^3 + \delta \mu )</td>
<td>( f_{yy} = 0 ) ( x_{\lambda\lambda} = 0 ) ( \eta = \text{sign}(x_{\lambda\lambda\lambda}) ) ( \varepsilon = \text{sign}(f_y), \delta = \text{sign}(f_{\mu}) )</td>
</tr>
<tr>
<td><strong>Homeostasis-Hopf bifurcations</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(7) ( x(\lambda) = \eta \lambda^2 )</td>
<td>( f(y, \mu) = (\varepsilon y^2 + \delta \mu)y )</td>
<td>( \eta = \text{sign}(x_{\lambda\lambda}) ) ( \varepsilon = \text{sign}(\rho_u), \delta = \text{sign}(\rho_\mu) )</td>
</tr>
<tr>
<td>(8) ( x(\lambda) = \eta \lambda^2 )</td>
<td>( f(y, \mu) = (\varepsilon y^2 + \delta \mu^2)y )</td>
<td>( \rho_\mu = 0 ) ( \eta = \text{sign}(x_{\lambda\lambda}) ) ( \varepsilon = \text{sign}(\rho_u), \delta = \text{sign}(\rho_{\mu\mu}) )</td>
</tr>
<tr>
<td>(9) ( x(\lambda) = \eta \lambda^2 )</td>
<td>( f(y, \mu) = (\varepsilon y^4 + \delta \mu)y )</td>
<td>( \rho_u = 0 ) ( \eta = \text{sign}(x_{\lambda\lambda}) ) ( \varepsilon = \text{sign}(\rho_{uu}), \delta = \text{sign}(\rho_\mu) )</td>
</tr>
<tr>
<td>(10) ( x(\lambda) = \eta \lambda^3 )</td>
<td>( f(y, \mu) = (\varepsilon y^3 + \delta \mu)y )</td>
<td>( x_{\lambda\lambda} = 0 ) ( \eta = \text{sign}(x_{\lambda\lambda\lambda}) ) ( \varepsilon = \text{sign}(\rho_u), \delta = \text{sign}(\rho_\mu) )</td>
</tr>
<tr>
<td>(11) ( x(\lambda) = \eta \lambda^2 )</td>
<td>( f(y, \mu) = (\varepsilon y^2 + \delta \mu^2)y )</td>
<td>( x_{\lambda\lambda} = 0 ) ( \rho_\mu = 0 ) ( \eta = \text{sign}(x_{\lambda\lambda\lambda}) ) ( \varepsilon = \text{sign}(\rho_u), \delta = \text{sign}(\rho_{\mu\mu}) )</td>
</tr>
<tr>
<td>(12) ( x(\lambda) = \eta \lambda^2 )</td>
<td>( f(y, \mu) = (\varepsilon y^3 + \delta \mu)y )</td>
<td>( x_{\lambda\lambda} = 0 ) ( \rho_u = 0 ) ( \eta = \text{sign}(x_{\lambda\lambda\lambda}) ) ( \varepsilon = \text{sign}(\rho_{uu}), \delta = \text{sign}(\rho_\mu) )</td>
</tr>
</tbody>
</table>

For Hopf bifurcations \( \rho(u, \mu) \) is defined by \( f(y, \mu) = \rho(y^2, \mu)y \). The numbers link information between tables.

* We always assume \( x_\lambda = 0 \) and \( f = f_y = 0 \). For Hopf bifurcations, we assume \( \rho = 0 \).
Table 4. Universal unfoldings.

<table>
<thead>
<tr>
<th>Universal Unfolding</th>
<th>Unperturbed Diagrams ($\varepsilon = -1$, $\eta = 1$)</th>
<th>$\delta = 1$</th>
<th>$\delta = -1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x(\lambda) = \eta \lambda^2$</td>
<td>$f(y, \mu, c) = \varepsilon y^2 + \delta (\mu - c)$</td>
<td><img src="image1" alt="Diagram 1" /></td>
<td><img src="image2" alt="Diagram 1" /></td>
</tr>
<tr>
<td>$x(\lambda) = \eta \lambda^2$</td>
<td>$f(y, \mu, c) = \varepsilon (y^2 + \delta (\mu - c)^2 + b)$</td>
<td><img src="image3" alt="Diagram 2" /></td>
<td><img src="image4" alt="Diagram 2" /></td>
</tr>
<tr>
<td>$x(\lambda) = \eta \lambda^2$</td>
<td>$f(y, \mu, c) = \varepsilon y^3 + \delta (\mu - c) + by$</td>
<td><img src="image5" alt="Diagram 3" /></td>
<td><img src="image6" alt="Diagram 3" /></td>
</tr>
<tr>
<td>$x(\lambda) = \eta \lambda^3 + a \lambda$</td>
<td>$f(y, \mu, c) = \varepsilon y^2 + \delta (\mu - c)$</td>
<td><img src="image7" alt="Diagram 4" /></td>
<td><img src="image8" alt="Diagram 4" /></td>
</tr>
<tr>
<td>$x(\lambda) = \eta \lambda^3 + a \lambda$</td>
<td>$f(y, \mu, c) = \varepsilon (y^2 + \delta (\mu - c)^2 + b)$</td>
<td><img src="image9" alt="Diagram 5" /></td>
<td><img src="image10" alt="Diagram 5" /></td>
</tr>
<tr>
<td>$x(\lambda) = \eta \lambda^3 + a \lambda$</td>
<td>$f(y, \mu, c) = \varepsilon y^3 + \delta (\mu - c) + by$</td>
<td><img src="image11" alt="Diagram 6" /></td>
<td><img src="image12" alt="Diagram 6" /></td>
</tr>
</tbody>
</table>

Blue (red) curves indicate stable (unstable) equilibria. The numbers link information between tables.
Table 4. Universal unfoldings (continued).

<table>
<thead>
<tr>
<th>Universal Unfolding</th>
<th>Unperturbed Diagrams ($\varepsilon = -1$, $\eta = 1$)</th>
<th>$\delta = 1$</th>
<th>$\delta = -1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x(\lambda) = \eta\lambda^2$&lt;br&gt;$f(y, \mu, c) = \varepsilon y^3 + \delta(\mu-c)y$</td>
<td><img src="image1" alt="Diagram" /></td>
<td><img src="image2" alt="Diagram" /></td>
<td><img src="image3" alt="Diagram" /></td>
</tr>
<tr>
<td>$x(\lambda) = \eta\lambda^2$&lt;br&gt;$f(y, \mu, c) = \varepsilon y^3 + \delta(\mu-c)^2y + by$</td>
<td><img src="image4" alt="Diagram" /></td>
<td><img src="image5" alt="Diagram" /></td>
<td><img src="image6" alt="Diagram" /></td>
</tr>
<tr>
<td>$x(\lambda) = \eta\lambda^2$&lt;br&gt;$f(y, \mu, c) = \varepsilon y^5 + \delta(\mu-c)y + by^3$</td>
<td><img src="image7" alt="Diagram" /></td>
<td><img src="image8" alt="Diagram" /></td>
<td><img src="image9" alt="Diagram" /></td>
</tr>
<tr>
<td>$x(\lambda) = \eta\lambda^3 + a\lambda$&lt;br&gt;$f(y, \mu, c) = \varepsilon y^3 + \delta(\mu-c)y$</td>
<td><img src="image10" alt="Diagram" /></td>
<td><img src="image11" alt="Diagram" /></td>
<td><img src="image12" alt="Diagram" /></td>
</tr>
<tr>
<td>$x(\lambda) = \eta\lambda^3 + a\lambda$&lt;br&gt;$f(y, \mu, c) = \varepsilon y^3 + \delta(\mu-c)^2y + by$</td>
<td><img src="image13" alt="Diagram" /></td>
<td><img src="image14" alt="Diagram" /></td>
<td><img src="image15" alt="Diagram" /></td>
</tr>
<tr>
<td>$x(\lambda) = \eta\lambda^3 + a\lambda$&lt;br&gt;$f(y, \mu, c) = \varepsilon y^5 + \delta(\mu-c)y + by^3$</td>
<td><img src="image16" alt="Diagram" /></td>
<td><img src="image17" alt="Diagram" /></td>
<td><img src="image18" alt="Diagram" /></td>
</tr>
</tbody>
</table>

Blue (red) curves indicate stable (unstable) equilibria. Black curves indicate amplitudes of stable limit cycles. The numbers link information between tables.
Table 6. Persistent diagrams.

<table>
<thead>
<tr>
<th>Transition Variety Σ</th>
<th>Persistent Perturbations</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1)⁺ Simple homeostasis - limit point: (-y^2 + \lambda^2 - c)</td>
<td><img src="image1" alt="Diagram" /></td>
</tr>
<tr>
<td>(2)⁺ Simple homeostasis - isola: (-y^2 - (\lambda^2 - c)^2 + b)</td>
<td><img src="image2" alt="Diagram" /></td>
</tr>
</tbody>
</table>

Blue (red) curves indicate stable (unstable) equilibria. Homeostasis points are marked. Region (2)⁺(c) has a homeostatic plateau for which leaving the plateau is marked by a loss of steady-state.
Table 5. Transition varieties.

<table>
<thead>
<tr>
<th>Normal Form, $h$</th>
<th>$B$</th>
<th>$H$</th>
<th>$C$</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1)$^+$ $-y^2 + \lambda^2 - c$</td>
<td>${ c = 0 }$</td>
<td>0</td>
<td>$\emptyset$</td>
</tr>
<tr>
<td>(1)$^-$ $-y^2 - \lambda^2 + c$</td>
<td>${ c = 0 }$</td>
<td>0</td>
<td>$\emptyset$</td>
</tr>
<tr>
<td>(2)$^+$ $-y^2 - (\lambda^2 - c)^2 + b$</td>
<td>${ b = 0</td>
<td>c \geq 0 } \cup { b = c^2 }$</td>
<td>0</td>
</tr>
<tr>
<td>(2)$^-$ $-y^2 + (\lambda^2 - c)^2 + b$</td>
<td>${ b = 0</td>
<td>c \geq 0 } \cup { b = -c^2 }$</td>
<td>0</td>
</tr>
<tr>
<td>(3) $-y^3 + \lambda^2 - c + by$</td>
<td>${ -\theta \left( \frac{b}{3} \right)^2 + \theta b \left( \frac{b}{3} \right)^2 = c }$</td>
<td>$b = 0</td>
<td>c \geq 0$</td>
</tr>
<tr>
<td>(4) $-y^2 + \lambda^3 + a\lambda - c$</td>
<td>${ 2\theta \left( \frac{-a}{3} \right)^2 = c</td>
<td>a \leq 0 }$</td>
<td>0</td>
</tr>
<tr>
<td>(5)$^+$ $-y^2 + (\lambda^3 + a\lambda - c)^2 + b$</td>
<td>${ -(2\theta \left( \frac{-a}{3} \right)^2 - c)^2 = b</td>
<td>a \leq 0 }$</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>$\cup { b = 0 }$</td>
<td></td>
<td>$\cup { a = 0</td>
</tr>
<tr>
<td>(5)$^-$ $-y^2 - (\lambda^3 + a\lambda - c)^2 + b$</td>
<td>${ (2\theta \left( \frac{-a}{3} \right)^2 - c)^2 = b a \leq 0 }$</td>
<td>0</td>
<td>${ 2\theta \left( \frac{-a}{3} \right)^2 = c</td>
</tr>
<tr>
<td></td>
<td>$\cup { b = 0 }$</td>
<td></td>
<td>$\cup { a = 0</td>
</tr>
<tr>
<td>(6) $-y^3 + \lambda^2 a\lambda - c + by$</td>
<td>${ \theta_1 \left( \frac{-a}{3} \right)^2 + \theta_2 \left( \frac{b}{3} \right)^2 = 0</td>
<td>a \leq 0, \ b \geq 0 }$</td>
<td>$b = 0$</td>
</tr>
</tbody>
</table>

where $\theta = 1$ or $-1$, $D = \emptyset$ for each homeostasis-bifurcation point considered here. As in Table 4 we choose $\varepsilon = -1$ and $\eta = 1$. The numbers link information between tables with $(+)$ or $(-)$ indicating the sign of $\delta$ where appropriate. $\delta = 1$ otherwise.
Table 5. Transition varieties (continued).

<table>
<thead>
<tr>
<th>Normal Form, $h$</th>
<th>$B$</th>
<th>$H$</th>
<th>$C$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(7)^+$</td>
<td>$-y^3 + (\lambda^2 - c)y$</td>
<td>${c = 0}$</td>
<td>$\emptyset$</td>
</tr>
<tr>
<td>$(7)^-$</td>
<td>$-y^3 - (\lambda^2 - c)y$</td>
<td>${c = 0}$</td>
<td>$\emptyset$</td>
</tr>
<tr>
<td>$(8)^+$</td>
<td>$-y^3 + (\lambda^2 - c)^2y + by$</td>
<td>${b = 0 \mid c \geq 0} \cup {c^2 = -b}$</td>
<td>$\emptyset$</td>
</tr>
<tr>
<td>$(8)^-$</td>
<td>$-y^3 - (\lambda^2 - c)^2y + by$</td>
<td>${b = 0 \mid c \geq 0} \cup {c^2 = -b}$</td>
<td>$\emptyset$</td>
</tr>
<tr>
<td>$(9)^+$</td>
<td>$-y^5 + (\lambda^2 - c)y + by^3$</td>
<td>${b^2 = c} \cup {c = 0}$</td>
<td>$\emptyset$</td>
</tr>
<tr>
<td>$(9)^-$</td>
<td>$-y^5 - (\lambda^2 - c)y + by^3$</td>
<td>${b^2 = -c} \cup {c = 0}$</td>
<td>$\emptyset$</td>
</tr>
<tr>
<td>$(10)$</td>
<td>$-y^3 + (\lambda^3 + a\lambda - c)y$</td>
<td>${2\theta \left(\frac{-a}{3}\right)^3 = c \mid a \leq 0}$</td>
<td>$\emptyset$</td>
</tr>
<tr>
<td>$(11)$</td>
<td>$-y^3 + (\lambda^3 + a\lambda - c)^2y + by$</td>
<td>${2\theta \left(\frac{-a}{3}\right)^3 - c^2 = b \mid a \leq 0}$ \cup ${b = 0}$</td>
<td>$\emptyset$</td>
</tr>
<tr>
<td>$(12)$</td>
<td>$-y^5 + (\lambda^3 + a\lambda - c)y + by^3$</td>
<td>${(2^{\frac{5}{4}} - 1)\left(\frac{1}{2}\right)^{\frac{3}{4}} + 2\theta \left(\frac{-a}{3}\right)^3 = c \mid a \leq 0, b \geq 0}$ \cup ${2\theta \left(\frac{-a}{3}\right)^3 = c \mid a \leq 0}$</td>
<td>${b = 0}$</td>
</tr>
</tbody>
</table>

where $\theta = 1$ or $-1$. $\mathcal{D} = \emptyset$ for each homeostasis-bifurcation point considered here. As in Table 4 we choose $\epsilon = -1$ and $\eta = 1$. The numbers link information between tables with (+) or (−) indicating the sign of $\delta$ where appropriate. $\delta = 1$ otherwise.
Table 6. Persistent diagrams (continued).

<table>
<thead>
<tr>
<th>Transition Variety Σ</th>
<th>Persistent Perturbations</th>
</tr>
</thead>
</table>

(2) Simple homeostasis - simple bifurcation: $-y^2 + (\lambda^2 - c)^2 + b$

Blue (red) curves indicate stable (unstable) equilibria. Homeostasis points are marked. Region (2)(d) predicts a wide homeostatic plateau.

(3) Simple homeostasis - hysteresis: $-y^3 + \lambda^2 - c + by$

Blue (red) curves indicate stable (unstable) equilibria. Homeostasis points are marked. Region (2)(d) predicts a wide homeostatic plateau.
Table 6. Persistent diagrams (continued).

<table>
<thead>
<tr>
<th>Transition Variety Σ</th>
<th>Persistent Perturbations</th>
</tr>
</thead>
</table>

(4) Chair - limit point: $-y^2 + \lambda^3 + a\lambda - c$

(5) Chair - isola: $-y^2 - (\lambda^3 + a\lambda - c)^2 - b$

No solutions when $b > 0$.

Blue (red) curves indicate stable (unstable) equilibria. Homeostasis points are marked. (5)$^+$ has many regions which predict wide plateaus for which leaving the plateau is marked by a loss of steady-state. Region (5)$^+$ (e) is exceptionally homeostatic.
### Table 6. Persistent diagrams (continued).

<table>
<thead>
<tr>
<th>Transition Variety Σ</th>
<th>Persistent Perturbations</th>
</tr>
</thead>
<tbody>
<tr>
<td>(e)</td>
<td><img src="image" alt="Diagram e" /></td>
</tr>
<tr>
<td>(f)</td>
<td><img src="image" alt="Diagram f" /></td>
</tr>
<tr>
<td>(g)</td>
<td><img src="image" alt="Diagram g" /></td>
</tr>
<tr>
<td>(h)</td>
<td><img src="image" alt="Diagram h" /></td>
</tr>
<tr>
<td>(i)</td>
<td><img src="image" alt="Diagram i" /></td>
</tr>
</tbody>
</table>

(5)− Chair - simple bifurcation: $-y^2 + (\lambda^3 + a\lambda - c)^2 - b$

---

Blue (red) curves indicate stable (unstable) equilibria. Homeostasis points are marked. Region (5)− (b) predicts a particularly wide plateau. Regions (5)− (e) and (5)− (i) predict a plateau for which variation in one direction is marked by loss of steady-state.
Table 6. Persistent diagrams (continued).

<table>
<thead>
<tr>
<th>Transition Variety $\Sigma$</th>
<th>Persistent Perturbations</th>
</tr>
</thead>
<tbody>
<tr>
<td>(e)</td>
<td><img src="image_e" alt="Diagram" /></td>
</tr>
<tr>
<td>(f)</td>
<td><img src="image_f" alt="Diagram" /></td>
</tr>
<tr>
<td>(g)</td>
<td><img src="image_g" alt="Diagram" /></td>
</tr>
<tr>
<td>(h)</td>
<td><img src="image_h" alt="Diagram" /></td>
</tr>
<tr>
<td>(i)</td>
<td><img src="image_i" alt="Diagram" /></td>
</tr>
<tr>
<td>(j)</td>
<td><img src="image_j" alt="Diagram" /></td>
</tr>
</tbody>
</table>
Table 6. Persistent diagrams (continued).

<table>
<thead>
<tr>
<th>Transition Variety $\Sigma$</th>
<th>Persistent Perturbations</th>
</tr>
</thead>
<tbody>
<tr>
<td>(6) Chair - hysteresis: $-y^3 + \lambda^3 + a\lambda - c + by$</td>
<td></td>
</tr>
</tbody>
</table>

Blue (red) curves indicate stable (unstable) equilibria. Homeostasis points are marked. Regions (6)(c) and (6)(d) were highlighted in the introduction (figure 3).
Table 6. Persistent diagrams (continued).

<table>
<thead>
<tr>
<th>Transition Variety Σ</th>
<th>Persistent Perturbations</th>
</tr>
</thead>
<tbody>
<tr>
<td>(7)⁺ Simple homeostasis - simple Hopf: $-y^3 - (\lambda^2 - c)y$</td>
<td></td>
</tr>
</tbody>
</table>

(a) \hspace{1cm} (b)

(7)⁻ Simple homeostasis - simple Hopf: $-y^3 + (\lambda^2 - c)y$

(a) \hspace{1cm} (b)

(8)⁺ Simple homeostasis - isola Hopf: $-y^3 - (\lambda^2 - c)^2 y - by$

(a) \hspace{1cm} (b) \hspace{1cm} (c) \hspace{1cm} (d)

Blue (red) curves indicate stable (unstable) equilibria. Green curves indicate the minimum and maximum of stable limit cycles. Black curves indicate amplitude of stable limit cycles. Homeostasis points are marked.
Table 6. Persistent diagrams (continued).

<table>
<thead>
<tr>
<th>Transition Variety Σ</th>
<th>Persistent Perturbations</th>
</tr>
</thead>
<tbody>
<tr>
<td>(8) (-y^3 + (\lambda^2 - c)^2 y + by)</td>
<td></td>
</tr>
</tbody>
</table>

Blue (red) curves indicate stable (unstable) equilibria. Green curves indicate the minimum and maximum of stable limit cycles. Black curves indicate amplitude of stable limit cycles. Homeostasis points are marked. Regions (8)\(^+\) and (8)\(^-\) predict wide homeostatic plateaus in amplitude.
Table 6. Persistent diagrams (continued).

<table>
<thead>
<tr>
<th>Transition Variety Σ</th>
<th>Persistent Perturbations</th>
</tr>
</thead>
</table>

(10) Chair - simple Hopf: \(-y^3 + (\lambda^3 + a\lambda - c)y\)

(11) Chair - isola Hopf: \(-y^3 - (\lambda^3 + a\lambda - c)^2y - by\)

Blue (red) curves indicate stable (unstable) equilibria. Green curves indicate the minimum and maximum of stable limit cycles. Black curves indicate amplitude of stable limit cycles. Homeostasis points are marked. Regions \((11)^+(f)\), \((11)^+(h)\), and \((11)^+(d)\) were highlighted in the introduction (figure (5)).
Table 6. Persistent diagrams (continued).

<table>
<thead>
<tr>
<th>Transition Variety $\Sigma$</th>
<th>Persistent Perturbations</th>
</tr>
</thead>
<tbody>
<tr>
<td>$b &gt; 0$</td>
<td></td>
</tr>
<tr>
<td>(m)</td>
<td>(l)</td>
</tr>
</tbody>
</table>

(a) $\Sigma_1$  

(b) $\Sigma_2$  

(c) $\Sigma_3$  

(d) $\Sigma_4$  

(e) $\Sigma_5$  

(f) $\Sigma_6$  

(g) $\Sigma_7$  

(h) $\Sigma_8$  

(i) $\Sigma_9$  

(j) $\Sigma_{10}$  

(k) $\Sigma_{11}$  

(l) $\Sigma_{12}$  

(m) $\Sigma_{13}$  

(n) $\Sigma_{14}$  

(o) $\Sigma_{15}$  

(p) $\Sigma_{16}$  

(q) $\Sigma_{17}$  

(r) $\Sigma_{18}$  

(s) $\Sigma_{19}$  

(t) $\Sigma_{20}$  

(u) $\Sigma_{21}$  

(v) $\Sigma_{22}$  

(w) $\Sigma_{23}$  

(x) $\Sigma_{24}$  

(y) $\Sigma_{25}$  

(z) $\Sigma_{26}$  

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>$c$</td>
<td></td>
</tr>
</tbody>
</table>

Legend:
- $\lambda$: A point of interest.
- $\text{amp}$: An amplitude.

For more details on each diagram, please refer to the corresponding section in the original document.
Table 6. Persistent diagrams (continued).

<table>
<thead>
<tr>
<th>Transition Variety Σ</th>
<th>Persistent Perturbations</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(11)^{-} - y^3 + (\lambda^3 + a\lambda - c)^2 y + by$</td>
<td></td>
</tr>
</tbody>
</table>

Blue (red) curves indicate stable (unstable) equilibria. Green curves indicate the minimum and maximum of stable limit cycles. Black curves indicate amplitude of stable limit cycles. Homeostasis points are marked. Regions $(11)^{-}(a)$, $(11)^{-}(b)$, $(11)^{-}(c)$ predict wide homeostatic plateaus in amplitude and period.
Table 6. Persistent diagrams (continued).

<table>
<thead>
<tr>
<th>Transition Variety Σ</th>
<th>Persistent Perturbations</th>
</tr>
</thead>
<tbody>
<tr>
<td>(i)</td>
<td><img src="image1" alt="Diagram" /></td>
</tr>
<tr>
<td>(j)</td>
<td><img src="image2" alt="Diagram" /></td>
</tr>
<tr>
<td>(k)</td>
<td><img src="image3" alt="Diagram" /></td>
</tr>
</tbody>
</table>

(12) \(-y^5 + (\lambda^3 + a\lambda - c)y + by^3\)

Blue (red) curves indicate stable (unstable) equilibria. Green (orange) curves indicate the minimum and maximum of stable (unstable) limit cycles. Solid (dashed) black curves indicate amplitude of stable (unstable) limit cycles. Homeostasis points are marked. Region (12)(h) predicts coexistence of homeostatic steady-states and homeostatic limit-cycles. Varying \(\lambda\) will switch between the two types of solutions.
Table 6. Persistent diagrams (continued).

<table>
<thead>
<tr>
<th>Transition Variety Σ</th>
<th>Persistent Perturbations</th>
</tr>
</thead>
<tbody>
<tr>
<td>(e)</td>
<td><img src="image" alt="Diagram" /></td>
</tr>
<tr>
<td>(f)</td>
<td><img src="image" alt="Diagram" /></td>
</tr>
<tr>
<td>(g)</td>
<td><img src="image" alt="Diagram" /></td>
</tr>
<tr>
<td>(h)</td>
<td><img src="image" alt="Diagram" /></td>
</tr>
<tr>
<td>(i)</td>
<td><img src="image" alt="Diagram" /></td>
</tr>
<tr>
<td>(j)</td>
<td><img src="image" alt="Diagram" /></td>
</tr>
<tr>
<td>(k)</td>
<td><img src="image" alt="Diagram" /></td>
</tr>
</tbody>
</table>

Acknowledgments
We thank Mike Reed for many helpful conversations. The research of MG was supported in part by National Science Foundation Grant DMS-1440386 to the Mathematical Biosciences Institute.

Appendix A Reduction of system (4) to equation (5)
In the case of steady-state bifurcation we use the Lyapunov-Schmidt method to reduce (4) to a scalar equation. We show that the reduced equation preserves homeostasis points of a chosen state variable $Y_k$. 
Consider the equation
\[ F(Y, \mu, b) = 0 \]  
where \( Y \in \mathbb{R}^n \), \( \mu \in \mathbb{R} \) is a distinguished parameter, and \( b \in \mathbb{R}^p \) is a vector of auxiliary parameters. We begin by explaining what we mean by \( Y \) homeostasis points. Suppose that \( F = 0 \) and \( D_Y F \) is nonsingular at \((Y_0, \mu_0, b_0)\). Then we may apply the implicit function theorem to obtain a function \( Y(\mu, b) \in \mathbb{R}^n \) where, near \((\mu_0, b_0)\), \( F(Y(\mu, b), \mu, b) = 0 \). However, we will soon assume that \( F \) undergoes a bifurcation, so it is possible that there is another value of \( Y \), say \( \tilde{Y}_0 \), so that \( F(\tilde{Y}_0, \mu_0, b_0) = 0 \). For this reason it will be useful to specify the value of \( Y \) when describing the homeostasis point.

**Definition 5.1.** The function \( \rho : \mathbb{R} \times \mathbb{R} \times \mathbb{R}^p \to \mathbb{R} \) has a homeostasis point at \((\rho_0, \mu_0, b_0)\) if \( \rho(\mu_0, b_0) = \rho_0 \) and \( \rho_\mu(\mu_0, b_0) = 0 \).

**Derivation of the reduction**

Suppose \( F \) undergoes a simple 0 eigenvalue bifurcation at \((Y, \mu, b) = (0, 0, 0)\). We perform a Lyapunov-Schmidt reduction on \( F \) at the bifurcation point to get a scalar function, \( f : \mathbb{R} \times \mathbb{R} \times \mathbb{R}^p \to \mathbb{R} \) so that, locally, solutions to \( f(y, \mu, b) = 0 \) are in one-to-one correspondence with solutions of (16). Let \( L = (D_Y F)(0,0,0) \). Lyapunov-Schmidt reduction requires making a choice of complementary subspaces \( M \) to \( \ker(L) = \mathbb{R}\{v_0\} \) in \( \mathbb{R}^n \) and \( N \) to range\((L) = \mathbb{R}\{v_0^*\} \) in \( \mathbb{R}^n \). In order for Lyapunov-Schmidt reduction to preserve homeostasis of the \( k^{th} \) coordinate in the original equation \( F = 0 \) to the reduced equation \( f = 0 \), we must also assume the nondegeneracy condition on \( L \) at the origin

\[ \langle v_0, e_k \rangle \neq 0 \]

or \( e_k \notin (\ker L)^\perp \). It follows that we can split the domain of \( L \) via \( \mathbb{R}^n = \ker L \oplus M \) by choosing \( M = \text{span}\{e_i | i \neq k\} \). We additionally split the codomain of \( L \) via \( \mathbb{R}^n = \text{range} L \oplus N \). The choice of \( N \) is arbitrary.

Let \( E \) denote projection onto range \( L \) with \( \ker E = N \). We may solve \( F(Y, \mu, b) = 0 \) by simultaneously solving

\[
\begin{align*}
EF(Y, \mu, b) &= 0 \\
(I - E)F(Y, \mu, b) &= 0
\end{align*}
\]

where \( I \) denotes the \( n \times n \) identity.

The reduction continues by decomposing \( Y \) as \( Y = v + w \) where \( v \in \ker L \) and \( w \in M \). Applying the implicit function theorem to \( EF(v + w, \mu, b) = 0 \) yields \( w \equiv w(v, \mu, b) \). Define \( \phi : \ker L \times \mathbb{R} \times \mathbb{R}^p \to N \) by \( \phi(v, \mu, b) = (I - E)F(v + w(v, \mu, b), \mu, b) \). So that our reduction preserves stability, we require \( \langle v_0, v_0^* \rangle > 0 \), where \( \langle \cdot, \cdot \rangle \) is the standard inner product on \( \mathbb{R}^n \). This is called a consistent choice of \( v_0 \) and \( v_0^* \) (see [Golubitsky & Schaeffer, 1985] for more details). The reduction is now given by

\[ f(y, \mu, b) = \langle v_0^*, \phi(v_0, \mu, b) \rangle. \]  

(17)

Note that if \( (y, \mu, b) \) solves \( f(y, \mu, b) = 0 \) then \( Y \) in the corresponding solution of \( F(Y, \mu, b) = 0 \) can be recovered via

\[ Y = yv_0 + w(yv_0, \mu, b). \]  

(18)

**Preservation of the desired properties**

Solutions of \( f = 0 \) and \( F = 0 \) are in one-to-one correspondence so multiplicity of solutions is automatically preserved. Let \( \lambda_1, \ldots, \lambda_n \) denote the eigenvalues of \( D_Y F(0,0,0) \) with \( \lambda_1 = 0 \). Note that we have assumed \( \text{Re}(\lambda_i) \neq 0 \) for \( i \neq 1 \). The following proposition both shows that stability of solutions is preserved if we make an additional assumption on \( \lambda_i \).
Proposition 3. Suppose \( \text{Re}(\lambda_i) < 0 \) for \( i \neq 1 \). Then the equilibria of \( \dot{Y} = F(Y, \mu, b) \) corresponding to a solution, \((y, \mu, b)\), of \( f(y, \mu, b) = 0 \) is asymptotically stable if \( f_y(y, \mu, b) < 0 \) and unstable if \( f_y(y, \mu, b) > 0 \).

Proof. See chapter 1, Theorem 4.1 of [Golubitsky & Schaeffer, 1985].

Now, given \( f(y_0, \mu_0, b_0) = 0 \) and \( f_y(y_0, \mu_0, b_0) \neq 0 \) we may apply the implicit function theorem to obtain a curve \( y(\mu, b) \) where \( f(y(\mu, b), \mu, b) = 0 \) and \( y(\mu_0, b_0) = y_0 \). The one-to-one correspondence of \( f \) and \( F \) gives a corresponding curve of equilibria, \( Y(\mu, b) \) with \( Y(\mu_0, b_0) = Y_0 \).

Definition 5.2. If \( y \) has a homeostasis point at \((y_0, \mu_0, b_0)\) if and only if \( Y_k \) has a homeostasis point at \((Y_0)_k, \mu_0, b_0)\) where \( Y_0 \) is the corresponding solution to \( y_0 \), then \( f \) is \( Y_k \) homeostasis preserving.

Proposition 4. \( f \) is \( Y_k \) homeostasis preserving.

Proof. Suppose \( y(\mu, b) \) satisfies \( f(y(\mu, b), \mu, b) = 0 \). The corresponding solutions to \( F = 0 \) are given by (18):

\[
Y(\mu, b) = y(\mu, b)v_0 + w(yv_0, \mu, b).
\]

However, \( w(yv_0, \mu, b) \in M \) so \( w_k(y(\mu, b)v_0, \mu, b) \equiv 0 \) and we have \( Y_k(\mu, b) = y(\mu, b)(v_0)_k \). Therefore \( (Y_k)_k(\mu, b) = y_k(\mu, b)(v_0)_k \) and \( f \) preserves \( Y_k \) homeostasis.
References