Rigid Patterns of Synchrony for Equilibria and Periodic Cycles in Network Dynamics

Martin Golubitsky
Mathematical Biosciences Institute, Ohio State University, Columbus OH 43210, USA

Ian Stewart
Mathematics Institute, University of Warwick, Coventry CV4 7AL, UK
(Dated: February 2, 2016)

We survey general results relating patterns of synchrony to network architecture, applying the formalism of coupled cell systems. We also discuss patterns of phase-locking for periodic states, where nodes have identical waveforms but regularly spaced phases. We focus on rigid patterns, which are not changed by small perturbations of the differential equation. Symmetry is one mechanism that creates patterns of synchrony and phase-locking. In general networks there is another: balanced colorings of the cells. A symmetric network may have anomalous patterns of synchrony and phase-locking that are not consequences of symmetry. We introduce basic notions on coupled cell networks and their associated systems of admissible differential equations. Periodic states also possess spatio-temporal symmetries, leading to phase relations; these are classified by the $H/K$ Theorem and its analog for general networks.

I. INTRODUCTION, EXAMPLES, AND OVERVIEW

This paper discusses three separate but related issues: rigid patterns of synchrony for equilibria, rigid patterns of phase-shift synchrony for periodic cycles, and the relation of these ideas to network symmetry. Roughly speaking, the pattern of synchrony associated to a state $x = (x_1, \ldots, x_n)$ in an $n$-node network is a coloring of the $n$ nodes where two nodes $c, d$ have the same color only if $x_c = x_d$. The pattern of synchrony for an equilibrium is rigid if perturbing the network system moves the equilibrium, but does not change its pattern of synchrony. A $T$-periodic solution $x(t) = (x_1(t), \ldots, x_n(t))$ has phase-shift synchrony in nodes $c$ and $d$ if

$$x_d(t) = x_c(t + \theta_{cd} T)$$

where $0 \leq \theta_{cd} < 1$. The pattern of phase-shift synchrony for $x(t)$ is the set $\Theta = \{\theta_{cd}\}$. This pattern is rigid if perturbing the network system perturbs $x(t)$ but does not change $\Theta$. This paper reviews the classification of rigid patterns and the subtle relationships with 'balanced colorings' and 'network symmetries'.

Synchronous behavior in networks of dynamical systems (ODEs) has been widely studied. It occurs when observations of distinct nodes of the network lead to time series that closely resemble each other. The strongest form of resemblance occurs when the time series are identical; at the other extreme, they may be distinct but significantly correlated.

More generally, sets of nodes can synchronize in clusters: here observations of nodes in the same cluster give very similar time series, but time series from nodes in distinct clusters differ. A partition into synchronous clusters is a pattern of synchrony.

Here we give an overview of a general formalism for network dynamics, coupled cell systems, developed over the past decade, and describe some of the insights it provides into patterns of synchrony. It is well known [12, 26] that there is a close relationship between patterns of synchrony and network symmetries, because symmetry-breaking naturally creates patterns. More surprising is the occurrence of patterns of synchrony that have nothing to do with symmetry in the conventional group-theoretic sense [16, 19, 29].

Applications of symmetric coupled cell systems include animal locomotion [3, 5, 14, 15, 27, 28], binocular rivalry [6, 8], and the vestibular system [18, 24]. Applications of general coupled cell systems include binocular rivalry [7] and homeostasis [13].

We now give a series of examples to illustrate general phenomena that will be explained in more detail later.

![FIG. 1. Patterns of synchrony for equilibria in a 6-cell ring, induced by symmetry. The corresponding subgroup is indicated at the center of each ring. The trivial pattern (all cell states distinct) is not shown; it corresponds to the identity subgroup.](image-url)
Example I.1.

A bidirectional ring of identical cells with identical couplings is symmetric under the dihedral group $D_6$, the symmetry group of a hexagon. The methods of symmetric bifurcation theory [12, 17] show that there are many typical patterns of synchrony, associated with subgroups of the symmetry group. Figure 1 shows the non-trivial types of pattern; we explain their derivation in Section III. Here nodes with the same color are in the same state, a convention we use throughout.

For simplicity, the network illustrated has nearest-neighbor coupling, but the same catalog of patterns occurs if longer-range couplings are permitted, provided they preserve the $D_6$ symmetry.

Example I.2. In periodic dynamics there is a further related phenomenon: cell states may be phase-locked. They then have the same waveform, but with phase shifts. Figure 2 (left) shows a ring of three identical FitzHugh-Nagumo neurons with unidirectional coupling; Figure 2 (right) shows a typical periodic state in which all nodes have the same waveform, but successive waves are phase-shifted by one third of a period. See Section III.

Example I.3. Symmetry of the network is a common source of patterns of synchrony, but not the only one. For example, the network of Figure 3 has only trivial symmetry, but we prove in Section III that the synchrony pattern shown by the different colors can occur for both equilibria and periodic states. Moreover, for suitable ODEs there exist periodic states in which cells of the three colors have the same waveform, but with phase shifts $0, \frac{1}{3}, \frac{2}{3}$ of the period, much as in Figure 2 (right).

Section II introduces the setting of coupled cell networks and admissible ODEs, explained informally using examples. The interplay between admissible maps and patterns of synchrony motivates the concept of a balanced coloring, and key properties are stated, in particular the construction of a quotient network, in which synchronous cells are identified in a manner that preserves their dynamics.

Section III outlines fundamental ideas from symmetric dynamics, specializing them to symmetric networks. Subgroups of the symmetry group create natural invariant subspaces that induce patterns of synchrony. We discuss spatio-temporal symmetries of periodic states, and their relation to two distinct symmetry groups: spatial (pointwise) and spatio-temporal (setwise) symmetries. We state the $H/K$ Theorem for general ODEs, which classifies possible spatio-temporal symmetries. This theorem must be modified in the context of networks, and the results are different when the state spaces of cells are multidimensional $\mathbb{R}^k$, one-dimensional $\mathbb{R}$, or the circle $\mathbb{S}^1$ (phase oscillators). We discuss multirhythms, a type of resonance caused by symmetry. Generalizations to arbitrary networks are stated.

Section IV gives examples of anomalous patterns of synchrony and phase shifts, existing in symmetric networks but not caused by symmetry.

Section V discusses general networks, where rigid patterns of synchrony—those preserved by small perturbations of the admissible ODE—correspond to balanced colorings, both for equilibria and periodic states. There is an analog of the $H/K$ Theorem for rigid patterns of phase shifts, which are induced by cyclic group symmetry on the quotient network defined by synchrony.

II. COUPLED CELL NETWORKS

First, we sketch the formalism of [19, 29]: coupled cell networks, with associated ODEs called coupled cell systems. Here the cells (nodes) of the network correspond to state variables and the arrows (directed edges) indicate couplings. Each arrow has a head (to which it points) and a tail cell. A key feature is that arrows are classified into one or more types. Arrows of the same type couple their inputs to their outputs in the same manner. There is also a notion of cell type: roughly speaking, cells with the same state space have the same type.

Associated with each such network is a class of admissible ODEs: those that respect the network structure and the arrow types. We introduce these ODEs below.

Formal definitions can be found in [19, 29], but here we work with typical examples, starting with Example I.3. Its network diagram has five cells and two types of arrow, shown with solid and dashed lines (and different arrowheads for emphasis). All cells have the same type, and are drawn as circles.
Bidirectional arrows, as in Figure 1, should be interpreted as shorthand for two identical arrows, one in each direction.

A network is path-connected or transitive if any cell is connected to any other by some directed path (chain of arrows linked head to tail). It is homogeneous if all cells have the same number of input arrows of each type.

**Admissible Maps**

We now associate with any network diagram a class of maps (or vector fields), said to be admissible, which respect the network architecture and determine a class of ODEs coupled together according to the network.

The state (or phase) space $P_c$ of each cell $c$ can be any manifold; common choices are $\mathbb{R}^k$ for some $k$, especially $k = 1$, or the circle $S^1$ for phase oscillators. For simplicity, assume $P_c = \mathbb{R}$. Assign a state variable $x_c \in \mathbb{R}$ to each cell $c$. By convention, the time-derivative $\dot{x}_c$ is some function of $x_c$, together with the tail cells $x_{di}$ of all arrows with head $c$. For example, cell 4 receives inputs from cells 1 (solid arrow) and 3 (dashed arrow). Therefore the corresponding component of a system of admissible ODEs takes the form

$$\dot{x}_4 = f(x_4, x_1, x_3) \quad (\text{II.1})$$

for an arbitrary (infinitely differentiable) function $f$. By convention, the internal state variable $x_4$ is written first.

Now consider another cell, say 1. This has internal variable $x_1$, and receives inputs from cells 4 (dashed arrow) and 5 (solid arrow). Therefore cell 1 has the same input type as cell 4: there is a one-to-one correspondence between the sets of all input arrows that preserves arrow type. In the formalism, this correspondence implies that the equation for $\dot{x}_1$ involves the same function $f$, with variables inserted in a manner that preserves arrow types. That is,

$$\dot{x}_1 = f(x_1, x_5, x_4)$$

with variables in the same order: internal, solid arrow, dashed arrow.

In this example all five cells have the same input type (one solid arrow, one dashed), so the same $f$ occurs for them all. The resulting system of ODEs is:

$$\begin{align*}
\dot{x}_1 &= f(x_1, x_5, x_4) \\
\dot{x}_2 &= f(x_2, x_1, x_3) \\
\dot{x}_3 &= f(x_3, x_2, x_1) \\
\dot{x}_4 &= f(x_4, x_1, x_3) \\
\dot{x}_5 &= f(x_5, x_4, x_1)
\end{align*}$$

The map $F : \mathbb{R}^5 \rightarrow \mathbb{R}^5$ whose components are these five functions defines a vector field on $\mathbb{R}^5$. We call $F$ an admissible vector field, or more briefly admissible map, for the given network. The associated system of ODEs is an admissible ODE.

The general recipe that associates to any network diagram a class of admissible maps works in exactly the same way. Choose a state space $P_c$ for each cell $c$, with state variable $x_c \in P_c$. Use the set of input arrows to $c$ to define a component $\dot{x}_c = f_c(x_c, x_{T(c)})$ of a system of ODEs, where $x_{T(c)}$ is the vector of state variables for the tail cells of the input arrows. If cells $c,d$ have the same input type, require $f_d$ to be equal to $f_c$, provided the input variables are arranged so that arrows of the same type correspond. These equalities of functions imply various equalities among the cell state spaces $P_c$, induced by input-equivalences. Cells with the same state space are said to have the same cell type.

Suppose that cell $c$ receives two input arrows of the same type, with tail cells $i,j$. The recipe for inserting variables into $f_c$ can lead equally well to either

$$f_c(x_c, \ldots, x_i, x_j, \ldots) \quad \text{or} \quad f_c(x_c, \ldots, x_j, x_i, \ldots)$$

To resolve this ambiguity, we require both of the above expressions to be equal. That is: if $c$ has two (or more) inputs of the same arrow type, $f_c$ is symmetric in the corresponding input variables. This is often shown by putting a bar over the variables concerned. For example, the bidirectional ring of Figure 1 has admissible ODEs of the form

$$\dot{x}_c = f_c(x_c, x_{c-1}, x_{c+1})$$

with subscripts taken (mod 6).

Three further features should be noted:

(1) Arrows can form self-loops with the same head and tail cell.

(2) Several arrows may have the same head and the same tail.

(3) The state variable $x_c$ of each cell $c$ is assumed to appear as a distinguished variable in the corresponding $f_c$. If there are self-loops, it may appear elsewhere too.

We refer to these conventions as the multiarrow formalism. Although multiple arrows and self-loops may appear artificial, this convention can be motivated both by applications and by the internal requirements of the theory [19]. Convention (3) was originally assumed for pragmatic reasons: it makes bifurcation theory simpler because the identity map is admissible.

**Remark II.1.** In this set-up individual arrows do not correspond to specific terms in the ODE. This differs from other approaches, such as networks of nonlinear oscillators with linear coupling [23, 26], where (II.1) would be replaced by

$$\dot{x}_4 = f(x_4) + ax_1 + bx_3$$

Instead, arrows specify which variables occur (perhaps nonlinearly) in which components of the ODE, and encode (via input sets of arrows) when components of the
ODE involve the same function. They also encode symmetries of those functions, induced by permuting input arrows of the same type.

Special assumptions about the form of \( F \) are consistent with the coupled cell formalism, but introduce extra structure.

**A. Synchrony and Balanced Colorings**

Throughout this paper we employ a strong notion of synchrony: time-series are identical. Weaker notions of synchrony can also be considered in the coupled cell formalism, but we focus here only on the stronger notion.

**Definition II.2.** Let \( x(t) \) be a solution of an admissible ODE. Cells \( c, d \) are **synchronous** on \( x(t) \) if

\[
x_c(t) \equiv x_d(t) \quad \forall t \in \mathbb{R}
\]

A **pattern of synchrony** is a partition of the cells into disjoint subsets, such that cells are synchronous on \( x(t) \) if and only if they belong to the same part. (Alternatively, the pattern of synchrony is the equivalence relation corresponding to the partition.) For example, in Figure 3 the pattern of synchrony is \( \{1\}, \{2, 4\}, \{3, 5\} \).

Diagrammatically, a pattern of synchrony is often represented by coloring the cells so that synchronous cells have the same color.

We now explain why the pattern of synchrony in figure 3 is natural and typical. Suppose it occurs for some solution \( x(t) \) of the admissible ODE. Then we must have

\[
x_2(t) \equiv x_4(t) \quad x_3(t) \equiv x_5(t) \quad \forall t \in \mathbb{R}
\]

Let \( y_1 = x_1, y_2 = x_2 = x_4, y_3 = x_3 = x_5 \), and substitute to obtain the **restricted ODE**:

\[
\begin{align*}
\dot{y}_1 &= f(y_1, y_3, y_2) \\
\dot{y}_2 &= f(y_2, y_1, y_3) \\
\dot{y}_3 &= f(y_3, y_2, y_1)
\end{align*}
\]

(II.2)

This system appears overdetermined (five equations in three unknowns). However, the two equations for \( \dot{y}_2 \) are identical, and so are those for \( \dot{y}_3 \). So we are left with a well-defined reduced system:

\[
\begin{align*}
\dot{y}_1 &= f(y_1, y_3, y_2) \\
\dot{y}_2 &= f(y_2, y_1, y_3) \\
\dot{y}_3 &= f(y_3, y_2, y_1)
\end{align*}
\]

(II.3)

Any solution \( y(t) = (y_1(t), y_2(t), y_3(t)) \) of the reduced system can be **lifted** to yield a solution

\[
x(t) = (x_1(t), x_2(t), x_3(t), x_2(t), x_3(t))
\]

of the original ODE, and this has the pattern of synchrony \( \{1\}, \{2, 4\}, \{3, 5\} \). This is the sense in which that pattern of synchrony is natural and typical.

The corresponding **synchrony subspace**

\[
\Delta = \{ x \in \mathbb{R}^5 : x_2 = x_4, x_3 = x_5 \}
\]

is an invariant subspace for any admissible map \( F \); that is, \( F(\Delta) \subseteq \Delta \). The reduced equation is defined by the restriction of \( F \) to \( \Delta \), projected onto \( \mathbb{R}^3 \) by mapping \( (x_1, x_2, x_3, x_4, x_5) \) to \( (x_1, x_2, x_3) \). (We renamed this as \( (y_1, y_2, y_3) \) for clarity.)

Self-consistency of the restricted ODE, hence existence of the reduced ODE, is not automatic for an arbitrary pattern of synchrony. For example, suppose we instead work with the partition \( \{1, 2\}, \{3\}, \{4, 5\} \). The first two components of the the restricted ODE (II.2) become

\[
\begin{align*}
\dot{y}_1 &= f(y_1, y_4, y_3) \\
\dot{y}_3 &= f(y_1, y_1, y_3)
\end{align*}
\]

which are contradictory.

In [29], extended to the multarrow formalism in [19], it is proved that the following are equivalent:

1. The restricted ODE is consistent, so the reduced ODE is well-defined, for all admissible \( F \).
2. The synchrony space \( \Delta \) is invariant under all admissible \( F \).
3. The pattern of synchrony is a **balanced coloring**. That is, if cells \( c, d \) have the same color, their input sets correspond in a manner that preserves arrow types and assigns the same colors to corresponding tail cells.

This theorem classifies all possible patterns of synchrony that are robust—common to all admissible maps. Each pattern exists in any network, for any ‘generic’ admissible map: solve the reduced ODE and lift.

In later sections we show that balanced colorings also classify typical patterns of synchrony for equilibria and for periodic states, where ‘typical’ means that the patterns are structurally stable. That is, they persist if \( F \) is perturbed slightly (while remaining admissible).

The underlying intuition here is as follows. Suppose that, by any sensible definition of synchrony, a pattern of synchrony occurs that is not ‘accidental’, that is, does not depend on very special choices of the admissible map. Then corresponding input cells of synchronous cells, compared in suitable pairs, are themselves synchronous. More succinctly: synchronous cells have synchronous input patterns. If not, differences in inputs would push them away from synchrony. This intuition is good motivation, but it is insufficient to prove theorems about patterns of synchrony.

**B. Quotient Networks**

As \( f \) varies over all functions, the reduced system (II.3) gives precisely the admissible ODEs for another network, with three cells (one for each color), as in Figure 4.
Consider the corresponding equivariant ODE precisely the condition that this can be done consistently.

For each of these cells, copy its input set from the original network, matching all colors of tail cells. Balance is valid for any balanced coloring. The admissible maps are precisely the restrictions of the original maps—more precisely, projections of restrictions in which redundant coordinates are removed.

This construction, the formation of a quotient network, is valid for any balanced coloring. The admissible maps for the quotient network are precisely the restrictions of the original maps—more specifically, projections of restrictions in which redundant coordinates are removed.

III. SYMMETRY

Patterns of synchrony arise naturally when the network has symmetry. In this section we establish some basic results and give illustrative examples. We consider both equilibria and periodic states, and describe patterns of phase shifts as well as synchrony.

Equiinviant ODEs

Before specializing to networks, we recall some basic concepts from symmetric (‘equivariant’) dynamics [12, 17]. Let $\Gamma$ be a finite group acting linearly on a finite-dimensional vector space $V$. A map $F : V \to V$ is $\Gamma$-equivariant if

$$F(\gamma x) = \gamma F(x) \quad \forall x \in V$$

Consider the corresponding equivariant ODE

$$\frac{dx}{dt} = F(x) \quad (\text{III.1})$$

If $x(t)$ is a solution, so is $\gamma x(t)$ for any $\gamma \in \Gamma$:

$$\frac{d}{dt} \gamma x(t) = \gamma \frac{dx}{dt} = \gamma F(x(t)) = F(\gamma x(t)) \quad (\text{III.2})$$

Recall that an orbit of $\Gamma$ on $V$ is a set of the form $\{\gamma v_0 : \gamma \in \Gamma\}$ for some choice of $v_0$. In this language, equation (III.2) states that solutions occur as group orbits. Conversely, this property of solutions implies equivariance of $F$.

Let $x(t) \equiv x_0$ be an equilibrium. Then $\gamma x_0$ is also an equilibrium, but need not equal $x_0$. We therefore define the isotropy subgroup of $x_0$ to be

$$\Gamma_{x_0} = \{ \gamma \in \Gamma : \gamma x_0 = x_0 \}$$

This is the symmetry group of $x_0$. We say that $x_0$ breaks symmetry from $\Gamma$ to $\Gamma_{x_0}$. Conversely, given a subgroup $\Sigma \subseteq \Gamma$, we define its fixed-point subspace to be

$$\text{Fix}(\Sigma) = \{ x : \sigma x = x \ \forall \sigma \in \Sigma \}$$

More generally, we can define the isotropy subgroup of a solution $x(t)$ to be

$$\Gamma_{x(t)} = \{ \gamma \in \Gamma : \gamma x(t) = x(t) \ \forall t \in \mathbb{R} \}$$

A basic result is:

Theorem III.1. For any subgroup $\Sigma \subseteq \Gamma$, the space $\text{Fix}(\Sigma)$ is invariant under $F$.

Proof. Let $x \in \text{Fix}(\Sigma)$, and let $\sigma \in \Sigma$. Then

$$\sigma F(x) = F(\sigma x) = F(x)$$

so $F(x) \in \text{Fix}(\Sigma)$. $\Box$

Although the proof is trivial, the result provides a method for finding solutions $x(t)$ of the ODE that break symmetry to any given $\Sigma$. Namely: restrict the ODE to $\text{Fix}(\Sigma)$ and solve. Usually $\text{Fix}(\Sigma)$ is smaller than $V$, a bonus.

Solutions found by this method need not be asymptotically stable. Symmetry methods can be employed to simplify stability computations, but stability is a separate issue, not addressed in this paper.

Network Symmetries

We now specialize the equivariant theory to symmetric networks. Most of the equivariant results still apply when the system is admissible for a $\Gamma$-symmetric network, where now $\Gamma$ is a permutation group acting on the cells. In a symmetric network, all admissible maps are equivariant, but equivariant maps need not be admissible because admissible maps satisfy extra constraints [1, 2].

The symmetry group of a network is the group $\Gamma$ of all permutations $\gamma$ of its cells that preserve the arrows, in the sense that for any pair of cells $c, d$ the number of arrows from $c$ to $d$ of a given type is the same as the corresponding number for arrows from $\gamma(c)$ to $\gamma(d)$. In particular, symmetries preserve input-type.

Orbits for permutation actions are defined to be sets of the form $\{ \gamma(c) : \gamma \in \Gamma \}$ for some choice of cell $c$. The orbits partition the set of cells. The permutation action of $\Gamma$ on cells induces an action on state variables by $x_c \mapsto x_{\gamma^{-1}(c)}$. 
Theorem III.2. Consider a network with symmetry group $\Gamma$, and any subgroup $\Sigma \subseteq \Gamma$. Then a solution $x(t)$ that lies in $\text{Fix}(\Sigma)$ has the pattern of synchrony determined by the orbits on $\Sigma$ on cells.

Proof. If $c \in \text{Fix}(\Sigma)$ then $c = \sigma(c)$ for all $\sigma \in \Sigma$, hence also $c = \sigma^{-1}(c)$ since also $\sigma^{-1} \in \Sigma$. Thus

$$x_{\sigma(c)}(t) \equiv x_c(t) \quad \forall t \in \mathbb{R}$$

so $x(t)$ is synchronous on each orbit of $\Sigma$. \hfill $\square$

Such a pattern is automatically balanced, but this can also be proved directly.

Figure 1 is an example. The choice of $\Sigma$ is shown at the center of each picture and the corresponding pattern of synchrony is shown by shading synchronous cells identically. The notation for the subgroups is determined by:

$$\pi = (14)(25)(36)$$
$$\kappa = (26)(34)$$
$$\zeta = (123456)$$

Generators are included in parentheses (for example $\mathbb{Z}_2(\pi)$) where the conjugacy class of the subgroup is ambiguous; otherwise the symbol indicates the isomorphism type of the subgroup. For example, $\mathbb{Z}_3$ is the cyclic subgroup of order 3 generated by $\zeta^2$. Notice that different subgroups may generate the same pattern.

Spatio-Temporal Symmetries

The above method applies to synchronous solutions of any type: equilibrium, periodic, even chaotic. But it addresses only spatial (pointwise) symmetries of the solution: those valid for all time. Periodic states also have temporal symmetries; for example, by definition a periodic state is symmetric under phase shifts that are integer multiples of the period. Combining the two leads to spatio-temporal symmetries, which are fundamental for periodic states.

Example III.3. We discuss Example I.2 in more detail. The time series in Figure 2 (right) are generated by a model in which each cell is a FitzHugh-Nagumo neuron, with voltage-coupling. Let $v_i$ denote the membrane potential of cell $i$, let $w_i$ be a surrogate for an ionic current, and suppose that $a, b, \gamma$ are parameters with $0 < a < 1$, $b > 0, \gamma > 0$. The coupling adds a voltage term to each cell equation, giving the system:

$$\dot{v}_1 = v_1(a - v_1)(v_1 - 1) - w_1 - cv_2$$
$$\dot{w}_1 = bv_1 - \gamma w_1$$
$$\dot{v}_2 = v_2(a - v_2)(v_2 - 1) - w_2 - cv_3$$
$$\dot{w}_2 = bv_2 - \gamma w_2$$
$$\dot{v}_3 = v_3(a - v_3)(v_3 - 1) - w_3 - cv_1$$
$$\dot{w}_3 = bv_3 - \gamma w_3$$

(III.3)

The symmetry group of the network is the cyclic group $\mathbb{Z}_3$ generated by the 3-cycle $(123)$, acting on pairs $(v_j, w_j)$ by permuting the indices cyclically.

When $a = b = \gamma = 0.5$ and $c = 0.8$, a numerical simulation shows that the origin is a stable equilibrium for the full six-dimensional system. In this state, the cells are synchronous. In contrast, when $a = b = \gamma = 0.5$ and $c = 2$, the system has no stable equilibrium (synchronous or not). Instead, it is attracted to a stable periodic state in which successive cells are one third of a period out of phase, Figure 2 (right). This figure shows the pattern for the $v_j$; a similar phase-locked pattern occurs for the $w_j$. Such a pattern is called a discrete rotating wave [17], phase-shift synchrony [12], or rosette phase locking [20].

This periodic state has spatio-temporal symmetry. If the period is $T$, and we set $x(t) = (v(t), w(t))$, it satisfies the phase relationships

$$x_2(t) = x_1(t - T/3) \quad x_3(t) = x_1(t - 2T/3)$$

and the solution $x(t)$ is invariant if we permute the labels using the 3-cycle $\rho = (123)$ and shift phase by $T/3$. That is,

$$\rho x(t + T/3) = x(t)$$

Thus $x(t)$ is fixed by the element $(\rho, T/3) \in \Gamma \times \mathbb{S}_1$, where $\mathbb{S}_1$ is the circle group of phase shifts modulo the period [17].

Pointwise and Setwise Symmetries

Spatio-temporal symmetries can be defined in a similar manner for periodic states of any symmetric dynamical system; in particular, any admissible ODE for a symmetric network. However, there is an alternative approach, which generalizes to chaotic states too—indeed, to states given by any dynamical attractor, in any reasonable sense of that term.

Associated with any attractor $A$ are two subgroups of $\Gamma$, its setwise and pointwise symmetries:

$$H_A = \{ \gamma \in \Gamma : \gamma A = A \}$$
$$K_A = \{ \gamma \in \Gamma : \gamma|_A = \text{id} \}$$

Clearly $K_A \subseteq H_A$, so the quotient group $H_A/K_A$ is defined.

If $A$ is an equilibrium then it is a single point, so $H_A = K_A = \Sigma_A$ and $H_A/K_A = 1$. An obvious necessary and sufficient condition for a given $H_A$ to occur is that it should be an isotropy subgroup of $\Gamma$.

If $A$ is a periodic cycle, the quotient group $H_A/K_A$ can be identified with phase shifts. To see how, suppose that $x(t)$ is a solution of the ODE defining the periodic cycle. If $\gamma \in H_A$ then $\gamma x(t)$ is also a solution. But $\gamma$ fixes $A$ setwise, so $\{ \gamma x(t) : t \in \mathbb{R} \} = \{ x(t) : t \in \mathbb{R} \}$. This there exists $\phi \in \mathbb{S}_1$ such that $\gamma x(0) = x(\phi)$. Now

$$\gamma x(t - \phi) = x(t) \quad \forall t \in \mathbb{R}$$

because both sides satisfy the same ODE and have the same initial condition at $t = \phi$, so by uniqueness of solutions of ODEs they are identical. But this equation determines a spatio-temporal symmetry. Since $K_A$ acts
trivially, the action of $H_A$ factors through $K_A$ giving an action of $H_A/K_A$. If $H_A$ is finite (as it must be for networks) then $H_A/K_A$ is a finite subgroup of $S^1$, hence a cyclic group $Z_k$ for some $k$.

**Example III.4.** In Example I.2 $H_A = Z_3$ and $K_A = 1$. Let $v(t)$ be a $T$-periodic solution to (III.3) and that $ρ = (123)$ generates the group $H_A ≅ Z_3$. Then $ρv(t)$ is also a periodic trajectory, differing only by time-translation. That is, $ρv(t) = v(t + θ)$ for all $t$. Applying the permutation $ρ = (123)$ three times (which gives the identity) implies that $3θ ≡ 0 \pmod{T}$. Hence either $θ = 0, \frac{T}{3}, \frac{2T}{3}$. But $θ ≠ 0$ by assumption. The other two yield rotating waves, which travel in opposite directions in the sense that $\frac{dT}{3} = -\frac{T}{3}$ modulo $T$.

Figure 5 shows the trajectory of $v(t)$ in $R^3$, viewed from a point very close to the main diagonal. The periodic cycle is shaped like a curved equilateral triangle, revealing the setwise $Z_3$ symmetry.

![FIG. 5. Setwise $Z_3$ symmetry of periodic trajectory of three coupled Fitzhugh-Nagumo neurons.](image)

An important feature of symmetry-induced phase shifts in periodic states is **rigidity**: if the vector field is slightly perturbed, the phase shifts remain unchanged (as a fraction of the period) [12]. Rigidity is not typical of phase shifts in general dynamical systems.

**H/K Theorem**

A pattern of phase shifts is similar to a pattern of synchrony, but phases are also taken into account. Cells whose states are phase-related with phase difference $0$ (that is, cells in the same $K_A$-orbit) are synchronous. But other pairs of cells (those in the same $H_A$-orbit but not in the same $K_A$-orbit) have the same waveform but different phases. We do not formalize this notion here. The pair $(H_A, K_A)$ determines the pattern of phase shifts, up to a slight ambiguity in the assignment of a phase to a generator of the group. So it is natural to ask for a classification of the possible pairs $(H_A, K_A)$.

For general dynamical systems, this is provided by the **H/K Theorem** of [4]. Let $N_Γ(K)$ denote the normalizer of $K$ in $Γ$, let $Γ_x$ denote the isotropy subgroup of $x ∈ R^N$,

and define a subset of $R^N$ by

$$L_K = \bigcup_{γ ∈ K} \text{Fix}(γ) \cap \text{Fix}(K) = \{x ∈ \text{Fix}(K) : K ⊆ Γ_x\}$$

**Theorem III.5.** Let $Γ$ be a finite group acting on $R^N$ and let $H ⊇ K$ be subgroups of $Γ$. There exists a hyperbolic periodic solution for a $Γ$-equivariant vector field on $R^N$ with spatial symmetries $K$ and spatiotemporal symmetries $H$ if and only if

(a) $K$ is a normal subgroup of $H$ and $H/K$ is cyclic.

(b) $K$ is an isotropy subgroup of $Γ$.

(c) $\text{dim Fix}(K) ≥ 2$, and if $\text{dim Fix}(K) = 2$ then either $H = K$ or $H = N_Γ(K)$.

(d) $H$ fixes a connected component of $\text{Fix}(K) \setminus L_K$.

Part (d) arises because of certain geometric obstacles associated with $L_K$. The proof shows that in the ‘if’ part of the theorem, which asserts the existence of suitable periodic states, the periodic state concerned can be made asymptotically stable. This theorem does not assert that periodic states with given $H, K$ occur for any admissible map, which is obviously false. It provides a complete catalog of the combinations that exist for some admissible map.

Analogos of this theorem for symmetric networks are examined in [9], for three distinct cases, in which cell state spaces are: $R^k$ for any $k$ (coupled systems); $R$ (coupled equations); $S^1$ (coupled oscillators). The key point is that these classifications are different. In other words, the state spaces of cells affect which patterns of phase shifts are possible. The results can be summarized as follows:

**Systems:** (a, b) as above. Change (c) to ‘if $\text{dim Fix}(K) = 2$ then $H = K = Γ$.’

**Equations:** (a, b, d) as above. Change (c) to ‘$\text{dim Fix}(K) ≥ 2$, and if $\text{dim Fix}(K) = 2$ then $H = K$.’

**Oscillators:** (a, b, d) as above. Change (c) to ‘if $\text{dim Fix}(K) = 2$ then $H = K = Γ$; if $\text{dim Fix}(K) = 2$ then either $H = K$ or $H = N_Γ(K)$’. These differences have two causes. Network symmetry groups are permutation groups, with special features, and the topology of cell state spaces imposes geometric constraints. Again the proofs show that in the ‘if’ part of the theorem the periodic state concerned can be made asymptotically stable. A similar classification for coupled systems given by path-connected networks was proved earlier [21].

**Multirhythms**

Symmetry can sometimes force a kind of resonance, in which some cells oscillate with a frequency that is a rational multiple of that of other cells. The overall state is periodic, but individual cells have different periods. A
simple example arises for the three-cell network of Figure 6. This has $\mathbb{Z}_2$ symmetry, generated by $\sigma = (13)$ which fixes cell 2. By the $H/K$ Theorem there can be a periodic state with $H = \mathbb{Z}_2, K = 1$. This has spatio-temporal symmetry generated by $(\sigma, T/2) \in \mathbb{Z} + 2 \times \mathbb{S}^1$: that is, swap cells using $\sigma$ and shift phase by half a period. This symmetry requires cells 1 and 3 to be half a period out of phase with each other; but because cell 2 is fixed, symmetry requires it to be half a period out of phase with itself. Therefore such states have the form

$$(x(t), y(t), x(t+T/2)) \quad y(t) = y(t+T/2)$$

implying that cell 2 oscillates with twice the frequency of cells 1 and 3. Such states can be found in simulations: we call them *multirhythms*. Unlike standard resonances, they arise because of symmetry. More elaborate examples are given in [12], showing in particular that it is possible for the minimal period of every cell to differ from that of the system as a whole; this occurs for example when symmetry leads to a 2:3 resonance.

![Figure 6. 3-cell network supporting a multirhythm state.](image)

**IV. ANOMALOUS PATTERNS**

When a network has symmetry, it is easy to imagine that its behavior is typical of symmetric dynamical systems with the same symmetry group. In fact, this assumption is often incorrect. If so, we call the behavior concerned *anomalous*. The reason is that although admissible maps are always equivariant, the converse can be false. In particular, equivariant maps need not depend on the correct input cells. Even when this restriction is added, admissible maps may obey extra constraints.

![Figure 7. Patterns of synchrony induced by symmetry (a) and by anomalous balanced colorings (b,c).](image)

Figure 7 is a case in point. It has three cells, and is homogeneous, with each cell receiving two inputs. (It is not path-connected, but that can be remedied by adding more arrows and the results do not change significantly.) The figure shows three 2-colorings. Each is balanced, and the corresponding quotient network is shown below the original.

The 3-cell network has $\mathbb{Z}_2$ symmetry, given by the permutation (12). The only nontrivial (not all identical or all different) pattern of synchrony predicted by the symmetry group is $\{1, 2\}, \{3\}$ as in (a). However, there are two more: $\{1, 3\}, \{2\}$ (b) and $\{1\}, \{2, 3\}$ (c). (To check balance in these two case, note that each grey cell receives one grey and one white input, and so does each white cell.) In case (a) the quotient is asymmetric. Both (b) and (c) have quotient networks with $\mathbb{Z}_2$ symmetry.

Consider periodic states, and to avoid dimension restrictions in the $H/K$ Theorem assume all cell phase spaces have dimension $\geq 2$. By the $H/K$ Theorem, the $\mathbb{Z}_2$ symmetry of the original network predicts a synchronized state of the form

$$(x(t), x(t), y(t))$$

and a multirhythm state of the form

$$(x(t), x(t+T/2), y(t)) \quad y(t+T/2) = y(t)$$

For the pattern of synchrony (b), the $\mathbb{Z}_2$ symmetry of the quotient predicts periodic states of the forms

$$(x(t), x(t), x(t)) \quad (x(t), y(t), x(t)) \quad (x(t), x(t+T/2), x(t))$$

In the third state, cells 1 and 2 are half a period out of phase; but instead of cell 3 having twice the frequency, it is synchronous with cell 1.

Pattern (c) gives similar results but with cells 1 and 2 interchanged.

**Example IV.1.** A pattern of synchrony in a symmetric network, not given by the orbits of a subgroup, is shown in Figure 8, and was discussed in [16]. Here the longer range arrows have the same type as the short range ones.

![Figure 8. 6-cell network with a balanced coloring not determined by symmetry.](image)

**Example IV.2.** Example IV.1, although a valid network, is slightly artificial because arrows unrelated by the symmetry group are given the same type. Figure 9 shows a pattern of synchrony on a 64-cell network, defined on a square lattice (mod 8). The symmetry group $\Gamma$ is generated by translation in both directions and the symmetries of the square. The figure shows only nearest-neighbor connections. However, if these are supplemented by longer-range connections, in any manner, subject only to the condition that all arrows in the same
Γ-orbit are included and have the same arrow-type, the pattern shown is balanced but is not the fixed-point set of a subgroup of Γ [1, 2].

FIG. 9. 64-cell network with a balanced coloring not determined by symmetry, even when extra group orbits of arrows are added. Arrows omitted; figure wraps round onto a torus.

FIG. 10. Left: Multiarrow network with \( \mathbb{Z}_3 \) symmetry. Right: Quotient with \( \mathbb{Z}_2 \) symmetry.

Example IV.3. The \( \mathbb{Z}_3 \) symmetry of the left-hand network of Figure 10 predicts, via the \( H/K \) Theorem, a rotating wave multirhythm of the form

\[
(x(t), x(t + T/3), x(t + 2T/3), y(t)) \quad (IV.1)
\]

where \( y(t) \) has triple the frequency of \( x(t) \). On the other hand, the 2-cell quotient predicts a pattern of the form

\[
(x(t), x(t + T/2), x(t + T/2), x(t + T/2)) \quad (IV.2)
\]

Both are possible with suitable state space dimensions and initial conditions. But only the first is evident from the symmetry of the left-hand network.

V. GENERAL NETWORKS

We now extend the main results about symmetric networks to general networks, again concentrating on equilibria and periodic states. Some analogous results are known, and others can plausibly be conjectured, for more complex dynamics, such as quasiperiodicity and chaos. We do not discuss these ideas here.

Rigidity

Virtually any partition of the cells of a network can be a pattern of synchrony by making a special choice of the admissible vector field. To avoid ‘accidental’ patterns of this kind, we require a form of structural stability. First, recall

**Definition V.1.** Consider an ODE

\[
\frac{dx}{dt} = F(x) \quad x \in \mathbb{R}^n \quad (V.1)
\]

(a) An equilibrium \( x_0 \) of (V.1) is hyperbolic if the Jacobian (derivative) of \( F \) has no eigenvalue on the imaginary axis.

(b) A periodic state \( x(t) \) of (V.1) of period \( T \) is hyperbolic if it has no Floquet exponent on the imaginary axis.

Hyperbolicity implies that after a small perturbation of the vector field, in case (a) there exists a unique equilibrium \( \tilde{x}_0 \) near \( x_0 \); while in case (b) there exists a unique \( T \)-periodic orbit \( \tilde{x}(t) \) near \( x(t) \) in the \( C^1 \) topology, with \( \tilde{T} \) near \( T \) [22]. Thus we can speak of ‘the perturbed equilibrium/periodic cycle’.

Now we can define the key concept of rigidity:

**Definition V.2.** Suppose \( F \) in (V.1) is admissible for a network.

(a) The pattern of synchrony for an equilibrium or periodic cycle of (V.1) is rigid if, for all sufficiently small perturbations, the perturbed equilibrium or periodic cycle has the same pattern of synchrony.

(b) A pattern of phase shifts for a periodic cycle of (V.1) is rigid if, for all sufficiently small perturbations, the perturbed periodic cycle has the same pattern of phase shifts (as fractions of the period).

Rigidity is not typical of phase shifts in general dynamical systems. However, in a symmetric ODE, the pattern of synchrony for an equilibrium or periodic cycle determined by the orbits of its isotropy subgroup is also rigid. The pattern of phase shifts for a periodic cycle determined by the \( H/K \) Theorem is rigid [10–12].

In the equilibrium case, we can classify rigid patterns of synchrony [19]:

**Theorem V.3.** Let \( x_0 \) be a hyperbolic equilibrium of an admissible ODE (V.1). Then the pattern of synchrony for \( x_0 \) is rigid if and only if it corresponds to a balanced coloring.

In the periodic case, we can classify rigid patterns of synchrony and rigid patterns of phase shifts, subject to a mild technical condition. This result combines the results of [10, 11] with a theorem of [25]. It is an analog for general networks of the \( H/K \) Theorem, in which \( K \) becomes a balanced coloring and \( H/K \) acts on the corresponding quotient network.

**Theorem V.4.** Let \( x(t) \) be a hyperbolic periodic cycle of an admissible ODE (V.1). Suppose that the network is path-connected, and all cell equivalent cells are input equivalent. Then:
(1) If the pattern of synchrony for \(x(t)\) is rigid, it corresponds to a balanced coloring.

(2) If the pattern of phase shifts for \(x(t)\) is rigid, then the quotient network corresponding to the pattern of synchrony has a cyclic symmetry group, and the pattern of phase shifts is that of a rotating wave induced by this symmetry group.

(3) Conversely, any pair consisting of a balanced coloring and a cyclic group of symmetries of the quotient network determines a pattern of synchrony and a pattern of phase shifts.

Part (2) is remarkable because it deduces a global conclusion, cyclic symmetry, from local comparisons of phase shifts. It is a consequence of the rigid phase shift theorem \([10, 11]\), which states that, subject to technical conditions, phase-shifted cells have inputs related by the same phase shift.

**Example V.5.** Consider the network of Figure 3 (left).

This has a rigid pattern of phase shifts of the form
\[
(x(t), x(t + T/3), x(t + 2T/3), x(t + T/3), x(t + 2T/3))
\]
(V.2)

On the quotient, the pattern of synchrony for this state is \(\{1\}, \{2, 4\}, \{3, 5\}\) which is balanced. The quotient network is shown in Figure 3 (right), and this has \(\mathbb{Z}_3\) symmetry. The pattern of phase shifts is that of a rotating wave with \((H, K) = (\mathbb{Z}_3, 1)\). This explains the periodic state described in Example I.3.

**Example V.6.** Different patterns of phase shifts on the same network may arise from different balanced colorings and different quotient networks. Figure 10 is an example. The pattern (IV.1) comes from the \(\mathbb{Z}_3\) symmetry of the 4-cell network, with a trivial pattern of synchrony whose quotient network is the same as the original. The pattern (IV.2) comes from the \(\mathbb{Z}_2\) symmetry of the 2-cell quotient network, corresponding to a non-trivial pattern of synchrony for the 4-cell network.

It is not possible to explain both patterns using the same symmetry group, because that would have to contain \(\mathbb{Z}_6\), which is not the symmetry group of any network with four or fewer cells.

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