Notes on Categorical Homotopy Theory

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Contents

Introduction

This document compiles the content of a reading course completed in Summer 2020 by the authors at the Ohio State University under Niles Johnson. It represents the authors first foray into any "serious" category theory, directly after finishing basic graduate courses in topology (from Hatcher) and algebra.

The course roughly followed the first 3 sections of Emily Riehl's *Categorical Homotopy Theory* [Rie14], with brief excursions into related topics, not all of which are reflected here. In any case, below are a selection of definitions, exercises, and (mostly) fleshed out proofs. As the creation of this document was entirely a learning experience, comments or corrections are always welcome!

Acknowledgements

Deep thanks to Niles for his time generously spent supervising the authors learning, as well as for his insightful commentary, helpful suggestions, and advice on navigating graduate school and beyond.

Chapter 1

Categorical Notions

1.1 Free - forgetful adjunction

Problem 1.1.1 ([Rie17, 4.2.iii])**.** Pick your favorite forgetful functor from Example 4.1.10 and prove that it is a right adjoint by defining its left adjoint, the unit, and the counit, and demonstrating that the triangle identities hold.

Proof. We choose the forgetful functor $U\colon {}_R\mathbf{Mod} \to \mathbf{Set}$ and show it is right adjoint to the free functor $F\colon \mathbf{Set} \to {}_R\mathbf{Mod}$. The unit η is a natural transformation $\mathrm{Id}_{\mathbf{Set}} \Rightarrow UF$, which we define $\eta_S(s) = s \in UF(S)$. The legs of the counit $\varepsilon\colon FU \Rightarrow \mathrm{Id}_{{}_R\mathbf{Mod}}$ are given by $\varepsilon_M(x) = x$ for $x \in M$. This makes sense since the elements of the module M are are a basis for $UF(M)$ by definition so the assignment on basis elements extends to a module homomorphism. We choose to omit the verification that η, ε are natural. The triangle identities to check are

$$\begin{array}{ccc} F & \xrightarrow{F\eta} & FUF \\ & \searrow^{\mathrm{Id}_F} & \Downarrow \varepsilon F \\ & & F \end{array} \qquad \text{and} \qquad \begin{array}{ccc} U & \xrightarrow{\eta U} & UFU \\ & \searrow^{\mathrm{Id}_U} & \Downarrow U\varepsilon \\ & & U \end{array}$$

We evaluate the nontrivial composites above. The primary difficulty is notational as the unit and counit are "identity" morphisms so in some sense the diagrams trivially commute. Anyway, the nontrivial composite on the left on a set S first sends $x \in F(S)$ to $x \in FUF(S)$ and then to $x \in F(S)$; this is the identity. The nontrivial composite on the right sends $s \in U(M)$ to $s \in UFU(M)$ and finally to $s \in U(M)$, which is also the identity. ■

1.2 The (co)End times

Definition 1.2.1 ([Ric20], 4.4.1)**.** Let $\mathcal{D}$ and $\mathcal{E}$ be categories. Let $H_1\colon \mathcal{D}^{op} \times \mathcal{D} \to \mathcal{E}$ and $H_2\colon \mathcal{D}^{op} \times \mathcal{D} \to \mathcal{E}$ be functors, and let

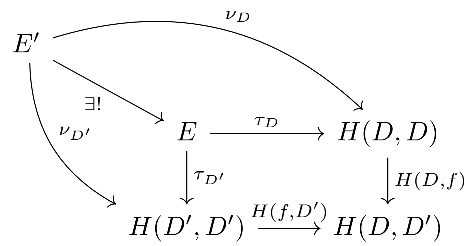
$$\tau_D\colon H_1(D,D) \to H_2(D,D)$$

be a family (indexed over objects in $\mathcal{D}$) of morphisms $\tau_D \in \mathcal{E}(H_1(D,D), H_2(D,D))$. Then $(\tau_D)_D$ is called a **dinatural transformation** (diagonal natural) if for all morphisms $f \in \mathcal{D}(D,D')$, the diagram

$$\begin{array}{ccccc} H_1(D',D) & \xrightarrow{H_1(f,D)} & H_1(D,D) & & \\ \downarrow H_1(D',f) & & & \searrow^{\tau_D} & \\ H_1(D',D') & & & & H_2(D,D) \\ & \searrow^{\tau_{D'}} & & & \downarrow H_2(D,f) \\ & & H_2(D',D') & \xrightarrow{H_2(f,D')} & H_2(D,D') \end{array}$$

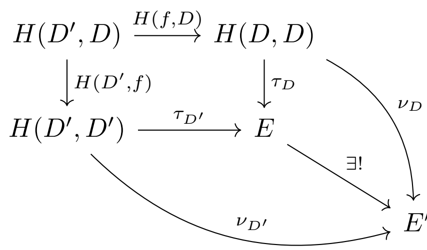
commutes.

Definition 1.2.2 ([Ric20], 4.4.4)**.** Let $H\colon \mathcal{D}^{op} \times \mathcal{D} \to \mathcal{E}$ be a functor. An **end** of H is a pair (E, τ) where E is an object of $\mathcal{E}$ and τ is a dinatural transformation from constant functor E to H which satisfy universal property showed the diagram below, for any other such pair (E', ν)



The object E is usually denoted by $\int_{\mathcal{D}} H$.

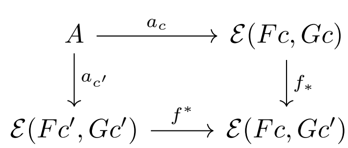
Definition 1.2.3 ([Ric20], 4.4.6)**.** The definition is dual to the definition of end. The universal diagram of the coend is as below.



The object E of $\mathcal{E}$ is usually denoted by $\int^{\mathcal{D}} H$.

Exercise 1.2.4 ([Rie14] 1.2.8)**.** Let $F, G : \mathcal{C} \rightrightarrows \mathcal{E}$, with $\mathcal{C}$ small and $\mathcal{E}$ locally small. Show that the end over $\mathcal{C}$ of the bifunctor $\mathcal{E}(F-, G-) : \mathcal{C}^{op} \times \mathcal{C}^{op} \to \mathbf{Set}$ is the set of the natural transformations from F to G.

Proof. $(A \subseteq N)$ Let (A, a) be end of the bifunctor $\mathcal{E}(F-, G-)$, where $A \in \mathbf{Set}$ and η is a dinatural transformation from $A \Rightarrow \mathcal{E}(F-, G-)$ (see 1.2.1). Then we have following commuting square.

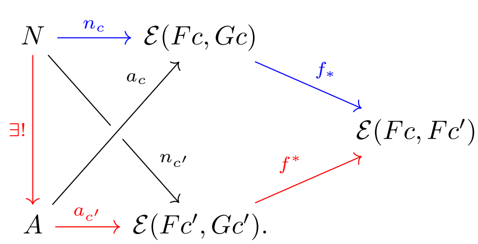


Now let N be the set of natural transformation from F to G, and let $\eta \in A$ with $\eta_c = a_c(\eta)$. Then η is a natural transformation, since for every $f : c \to c'$ from the commuting square we get,

$$Gf \circ \eta_c = \eta_{c'} \circ Ff.$$

Thus $A \subseteq N$.

$(A \supseteq N)$ Now consider a dinatural transformation $n : N \Rightarrow \mathcal{E}(F-, G-)$ given by $n_c(\eta) = \eta_c$. Since η is a natural transformation, it satisfies the above naturality equation for every $f : c \to c'$. From the universal property of ends, we get a unique transformation from $r : N \to A$.



Let $i : A \to N$ be inclusion. Let $\eta \in N$ and let $\eta' = r(\eta)$. Let $c \in \mathcal{C}$ be arbitrary. Consider $\text{Id}_c : c \to c$. Then chasing $\eta \in N$ through red (left-bottom) and blue (top) arrows we get,

$$\eta_c = \eta_c \circ \text{Id}_c = \text{Id}_c \circ r(\eta)_c.$$

Since c was arbitrary $\eta = r(\eta)$, and hence $N \subseteq A$, ergo $A = N$. ■

1.3 Whiskering

Definition 1.3.1. Let $G : \mathcal{C} \to \mathcal{D}$, $F, F' : \mathcal{D} \to \mathcal{M}$, and $K : \mathcal{M} \to \mathcal{N}$ be functors. Let $\eta : F \Rightarrow F'$ be natural transformation,

$$\mathcal{C} \xrightarrow{\;G\;} \mathcal{D} \underset{F'}{\overset{F}{\Downarrow \eta}} \mathcal{M} \xrightarrow{\;K\;} \mathcal{N}$$

then the **left whiskering** of η by G is a natural transformation $\eta G : \mathcal{C} \to \mathcal{M}$ between functors $F \circ G$ and $F' \circ G$, legs of which is given by $(\eta G)_c : = \eta_{Gc}$. Similarly the **right whiskering** of η by K is a natural transformation $K\eta : \mathcal{D} \to \mathcal{N}$ between functors $K \circ F$ and $K \circ F'$, legs of which is given by $(K\eta)_d : = K(\eta_d)$. naturality of ηG and $K\eta$ follows from naturality of η and functoriality of G and K respectively.

Definition 1.3.2. Let $\eta : F \to G$ and $\varepsilon : G \Rightarrow K$ be natural transformations,

$$\mathcal{C} \underset{K}{\overset{F}{\underset{\Downarrow \varepsilon}{\overset{\Downarrow \eta}{-\,G \to}}}} \mathcal{D} \; .$$

Then the composition $\varepsilon \circ \eta : F \to K$ is a natural transformation with is given by $(\varepsilon \circ \eta)_c = \varepsilon_c \circ \eta_c$. Naturality follows from the naturality of ε and η. We will drop the composition $\circ$ sign and write it as $(\varepsilon)(\eta)$.

Proposition 1.3.3. *We will list few properties of whiskering . Consider,*

$$\mathcal{M} \xrightarrow{\;G\;} \mathcal{C} \underset{F''}{\overset{F}{\underset{\Downarrow \varepsilon}{\overset{\Downarrow \eta}{-\,F' \to}}}} \mathcal{D} \xrightarrow{\;K\;} \mathcal{N} \; .$$

Then

1. $\big((\varepsilon)(\eta)\big)G = (\varepsilon G)(\eta G)$; $\quad K\big((\varepsilon)(\eta)\big) = (K\varepsilon)(K\eta)$.
2. $K(\eta G) = (K\eta)G$.
3. $(\eta G)G' = \eta(G \circ G')$ *for any* $G' : \mathcal{M}' \to \mathcal{M}$; $\quad K'(K\eta) = (K' \circ K)\eta$ *for any* $K' : \mathcal{N} \to \mathcal{N}'$.

Proof. We will check legs of each natural transformations

1. For $m \in \mathcal{M}$ we get,

$$\big[\big((\varepsilon)(\eta)\big)G\big]_m = \big((\varepsilon)(\eta)\big)_{Gm} = \varepsilon_{Gm} \circ \eta_{Gm} = (\varepsilon G)_m \circ (\eta G)_m = \big[(\varepsilon G)(\eta G)\big]_m$$

Thus $\big((\varepsilon)(\eta)\big)G = (\varepsilon G)(\eta G)$ and similarly we get $K\big((\varepsilon)(\eta)\big) = (K\varepsilon)(K\eta)$.

2. For $m \in \mathcal{M}$ we get,

$$\big[K(\eta G)\big]_m = K\big((\eta G)_m\big) = K\big(\eta_{Gm}\big) = (K\eta)_{Gm} = \big[(K\eta)G\big]_m$$

Thus $K(\eta G) = (K\eta)G$.

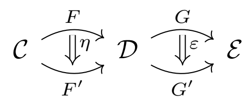
3. For $m' \in \mathcal{M}'$ we get,

$$\big[(\eta G)G'\big]_{m'} = (\eta G)_{G'm'} = \eta_{GG'm'} = \big[\eta(G \circ G')\big]'_{m}$$

Thus $(\eta G)G' = \eta(G \circ G')$ for any $G' : \mathcal{M}' \to \mathcal{M}$, similarly we get $K'(K\eta) = (K' \circ K)\eta$ for any $K' : \mathcal{N} \to \mathcal{N}'$.

■

Proposition 1.3.4. *Let $\eta : F \Rightarrow F'$ and $\varepsilon : G \Rightarrow G'$ be natural transformations,*



Then $(G'\eta)(\varepsilon F) = (\varepsilon F')(G\eta)$. This proposition says that if doing ε first and then η is same as doing η first and then ε.

Proof. We will check the legs of the transformations,

$$\begin{aligned}
\big((G'\eta)(\varepsilon F)\big)_c &= (G'\eta)_c \circ (\varepsilon F)_c \\
&= G'(\eta_c) \circ \varepsilon_{Fc} \\
&= \varepsilon_{F'c} \circ G(\eta_c) \\
&= (\varepsilon F')_c \circ (G\eta)_c \\
&= \big((\varepsilon F')(G\eta)\big)_c
\end{aligned}$$

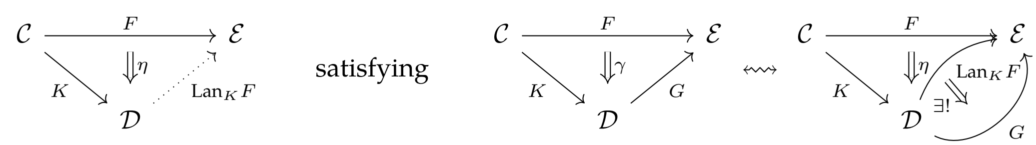
Thus $(G'\eta)(\varepsilon F) = (\varepsilon F')(G\eta)$. ■

Chapter 2

All Concepts Are Kan Extensions

2.1 Kan extensions

Definition 2.1.1 ([Rie14], 1.1.1)**.** Given functors $F\colon \mathcal{C} \to \mathcal{E}$, $K\colon \mathcal{C} \to \mathcal{D}$, a **left Kan extension** of F along K is a functor $\mathrm{Lan}_K F\colon \mathcal{D} \to \mathcal{E}$ together with a natural transformation $\eta\colon F \Rightarrow \mathrm{Lan}_K F \circ K$ such that for any other such pair $(G\colon \mathcal{D} \to \mathcal{E}, \gamma\colon F \Rightarrow G \circ K)$, γ factors uniquely through η. Diagramatically, we have



Remark 2.1.2 ([Rie14], 1.1.4). A left Kan extension of $F\colon \mathcal{C} \to \mathcal{E}$ along $K\colon \mathcal{C} \to \mathcal{D}$ is a representation for the functor

$$\mathcal{E}^{\mathcal{C}}(F, - \circ K)\colon \mathcal{E}^{\mathcal{D}} \to \mathbf{Set}$$

that sends a functor $\mathcal{D} \to \mathcal{E}$ to the set of natural transformations from F to its restriction along K. By the Yoneda lemma, any pair (G, γ) as in definition 2.1.1 defines a natural transformation

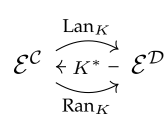
$$\mathcal{E}^{\mathcal{D}}(G, -) \xRightarrow{\gamma} \mathcal{E}^{\mathcal{C}}(F, - \circ K)$$

The universal property of the pair $(\mathrm{Lan}_k F, \eta)$ is equivalent to the assertion that the corresponding map

$$\mathcal{E}^{\mathcal{D}}(\mathrm{Lan}_K(F), -) \xRightarrow{\eta} \mathcal{E}^{\mathcal{C}}(F, - \circ K)$$

is a natural isomorphism, i.e., that $(\mathrm{Lan}_K(F), \eta)$ represents this functor. We conclude that if for a fixed K, the left and right Kan extensions of any functor exist, then these define left and right adjoints to the precomposition functor $K^*\colon \mathcal{E}^{\mathcal{D}} \to \mathcal{E}^{\mathcal{C}}$.

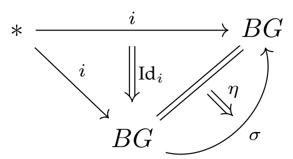
$$\mathcal{E}^{\mathcal{D}}(\mathrm{Lan}_K F, G) \cong \mathcal{E}^{\mathcal{C}}(F, GK)$$



$$\mathcal{E}^{\mathcal{C}}(GK, F) \cong \mathcal{E}^{\mathcal{D}}(G, \mathrm{Ran}_K F)$$

Exercise 2.1.3 ([Rie14], 1.1.3)**.** Construct a toy example to illustrate that if F factors through K along some functor H, it is not necessarily the case that $(H, 1_F)$ is the left Kan extension of F along K.

Proof. Let G be a group with an outer automorphism σ, such as $\mathbb{Z}/3\mathbb{Z}$ and the inversion map. We consider the following commutative diagram in **Cat**, where $i\colon * \to BG$ is the functor induced by the trivial group homomorphism:



Natural transformations $\eta : \mathrm{Id}_{BG} \to \sigma$ are in bijection with elements h of G such that $hx = \sigma(x)h$ for all $x \in G$, or equivalently $hxh^{-1} = \sigma(x)$. Since σ is outer, the element h does not exist, neither does η, and so $(\mathrm{Id}_{BG}, 1_i)$ is not Kan. ■

2.2 A formula for Kan extensions

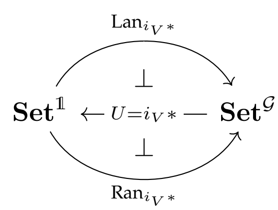
Theorem 2.2.1 ([Rie14], 1.2.1)**.** *When $\mathcal{C}$ is small, $\mathcal{D}$ is locally small, and $\mathcal{E}$ is co-complete, the left Kan extension of any functor $F : \mathcal{C} \to \mathcal{E}$ along any functor $K : \mathcal{C} \to \mathcal{D}$ is computed at $d \in \mathcal{D}$ by the colimit*

$$Lan_K F(d) \cong \int^{c \in C} \mathcal{D}(Kc, d) \cdot Fc \tag{2.2.1}$$

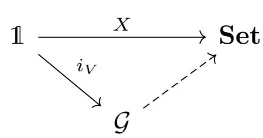
and in particular necessarily exists.

Exercise 2.2.2 ([Rie14], 1.2.11)**.** Directed graphs are functors from the category $\mathcal{G}$ with two objects E, V and a pair of maps $s, t : E \rightrightarrows V$ to $\mathbf{Set}$. A natural transformation between two such functors is a graph morphism. The forgetful functor $\mathbf{DirGph} \to \mathbf{Set}$ that maps a graph to its set of vertices is given by restricting along the functor from the terminal category $\mathbb{1}$ that picks out the object V. Use 2.2.1 to compute left and right adjoints to this forgetful functor.

Computation. For a functor $X : \mathbb{1} \to \mathbf{Set}$ we write $X(\bullet) = X$. Let i_V be the functor $\mathbb{1} \to \mathcal{G}$, picking out V. Then we have the following diagram by previous discussion (2.1.2):



So we can compute these left and right adjoints by computing the left and right Kan extensions which fill the dashed arrow in :



In particular for a functor $X : 1$ to $\mathbf{Set}$, there is only one object and one morphism in the domain, the corresponding coends and ends (2.2.1) simplify dramatically. For instance,

$$\mathrm{Lan}_{i_V} X(V) \cong \int^{x \in \mathbb{1}} \mathcal{G}(i_V(x), V) \cdot X(x) \cong \mathrm{coeq}\left(\coprod_{\mathrm{Id}_\bullet} \mathcal{G}(V, V) \cdot X \rightrightarrows \coprod_{\bullet} \mathcal{G}(V, V) \cdot X \right).$$

Here the top arrow in the last term induces the identity on X and the bottom is composition with $\mathrm{Id}_\bullet$ on $\mathcal{G}(V, V)$. Both of these are the identity map, and since $\mathcal{G}(V, V)$ has only one element, the coequalizer which gives the vertices of Lan_{i_V} is isomorphic to X. Similarly

$$\mathrm{Lan}_{i_V} X(E)) \cong \int^{x \in \mathbb{1}} \mathcal{G}(i_V(x), E) \cdot X(x) \cong \mathrm{coeq}\left(\coprod_{\mathrm{Id}_\bullet} \mathcal{G}(V, E) \cdot X \rightrightarrows \coprod_{\bullet} \mathcal{G}(V, E) \cdot X \right).$$

All the sets in this last diagram are empty as $\mathcal{G}(V, E) = \emptyset$, the coequalizer is as well, and so the set of edges is empty. In summary, the left adjoint to the forgetful functor assigns to a set X a graph with vertices X and no edges. The source and target morphisms s, t are forced to be empty, and the left adjoint is thus the **discrete graph** functor.
Dually, the right adjoint is computed by

$$\mathrm{Ran}_{i_V} X(V) \cong \int_{x \in \mathbb{1}} X(x)^{\mathcal{G}(V, i(V))} \cong \mathrm{eq}\left(X^{\{\mathrm{Id}_V\}} \rightrightarrows X^{\{\mathrm{Id}_V\}}\right) \cong X$$

where the last isomorphism comes since both the arrows in the equalizer diagram are the identity, one induced from the identity on $\mathrm{Id}_\bullet \in \mathbb{1}$ and one induced from Id_V. The computation of $\mathrm{Ran}_{i_V} X(E)$ is broadly similar:

$$\mathrm{Ran}_{i_V} X(E) \cong \int_{x \in \mathbb{1}} X(x)^{\mathcal{G}(E, i(V))} \cong \mathrm{eq}\left(X^{\mathcal{G}(E,V)} \rightrightarrows X^{\mathcal{G}(E,V)}\right) \cong X_s \times X_t,$$

as again both arrows are identities (induced from $\bullet$ and E). Both X_s and X_t are equal to X, we have merely labeled the two factors. There is no difference between s and t, so it remains to calculate the source arrow s. We claim $s \colon X_s \times X_t \to X$ is projection onto the X_s factor; by symmetry t will project onto the X_t factor. By definition, the map $\mathrm{Ran}_{i_V} X(s) \colon X^{\mathcal{G}(E,V)} \to X^{\mathcal{G}(V,V)}$ is induced by cotensoring with $s - \circ \colon \mathcal{G}(V,V) \to \mathcal{G}(E,V)$. This is the map $\mathbf{Set}(\{s,t\}, X) \to \mathbf{Set}(\{\mathrm{Id}_V\}, X)$ induced by precomposition with s, which projects onto the X_s factor. So the right adjoint to the forgetful functor takes a set X to the **chaotic graph** on X. This graph has vertices X and a unique directed arrow between every (ordered) pair of objects. ■

Exercise 2.2.3 ([Rie14], 1.4.8)**.** Use 2.2.1, the Yoneda Lemma, and the coYoneda lemma to deduce another form of the density theorem: that the left Kan extension of the Yoneda embedding $\Delta^\bullet \colon \mathcal{C} \to \mathbf{Set}^{\mathcal{C}^{\mathrm{op}}}$ along itself is the identity functor. This says that the representable functors form a **dense subcategory** of the presheaf category $\mathbf{Set}^{\mathcal{C}^{\mathrm{op}}}$.

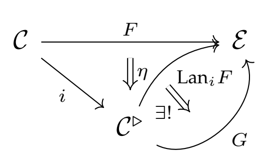
Proof. Let $F \in \mathbf{Set}^{\mathcal{C}^{\mathrm{op}}}$ and $c \in \mathcal{C}$. Then by 2.2.1, the Yoneda Lemma, the symmetry of the copower in **Set**, and finally the coYoneda lemma, we have (naturally in F, c):

$$\mathrm{Lan}_{\Delta^\bullet} \Delta^\bullet(F)(c) \cong \int^{x \in \mathcal{C}} \mathbf{Set}^{\mathcal{C}^{\mathrm{op}}}(\Delta^x, F) \cdot \Delta^x(c) \cong \int^{x \in \mathcal{C}} Fx \cdot \mathcal{C}(x, c) \cong \int^{x \in \mathcal{C}} \mathcal{C}(x,c) \cdot Fx \cong Fc\,.$$

Every isomorphism here is canonical, so it makes sense to say that $\mathrm{Lan}_{\Delta^\bullet} \Delta^\bullet(F)$ *is* F. ■

Exercise 2.2.4 ([Rie14] 1.2.7)**.** Let $\mathcal{C}$ be a small category and write $\mathcal{C}^{\triangleright}$ for the category obtained by adjoining a terminal object to $\mathcal{C}$. Give three proofs that a left Kan extension of a functor $F : \mathcal{C} \to \mathcal{E}$ along the natural inclusion $\mathcal{C} \to \mathcal{C}^{\triangleright}$ defines a colimit cone under F: one using the defining universal property, one using Theorem 1.2.1, and one using the formula of 1.2.6.

Proof. We have the following situation in general:



1. The defining universal property of the left Kan extension gives that for any functor G and natural transformation, $\gamma \colon F \Rightarrow G \circ i$, we get a unique natural transformation, $\mathrm{Lan}_i F \Rightarrow G$. We observe that the functor G gives the data of a cone under F: the image of the terminal object, $G(*)$ receives morphisms from the image of F by $F(c) \to Gi(c) \to G(*)$ where the first morphism is γ_c and the second morphism is the image of the unique arrow $c \to *$ in $\mathcal{C}^{\triangleright}$. In fact, any such cone takes this form, where the image of $*$ is the nadir of the cone: for a cone under F, set the functor F' to agree with F on all elements in $\mathcal{C}$ and send $F'(*)$ to the nadir of the cone. The natural transformation $F \Rightarrow Gi$ is simply the identity, and the functoriality of F' is ensured by the commutativity required by being a cone.

Let G be the functor corresponding to an arbitrary cone. $\mathrm{Lan}_i F$ also defines a cone under F and there exists a unique natural transformation, $\varepsilon : \mathrm{Lan}_i F \Rightarrow G$. Then the morphism ε_* gives a map of cones from that of $\mathrm{Lan}_i F$ to the cone defining G with the commutativity ensured by the naturality of ε. Since this morphism exists for any cone, we have that the cone under F defined by $\mathrm{Lan}_i F$ is a colimit cone.

2. Theorem 1.2.1 says that:

$$\mathrm{Lan}_i(F)(x) = \int^{c \in \mathcal{C}} \mathcal{C}^{\triangleright}(i(c), x) \cdot Fc := \mathrm{coeq} \left\{ \coprod_{f : c' \to c} \mathcal{C}^{\triangleright}(i(c), x) \cdot Fc' \rightrightarrows \coprod_{c \in \mathcal{C}} \mathcal{C}^{\triangleright}(i(c), x) \cdot Fc \right\}.$$

Specifically, evaluating at $*$, we get that:

$$\mathrm{Lan}_i(F)(*) = \mathrm{coeq} \left\{ \coprod_{f : c' \to c} \mathcal{C}^{\triangleright}(i(c), *) \cdot Fc' \rightrightarrows \coprod_{c \in \mathcal{C}} \mathcal{C}^{\triangleright}(i(c), *) \cdot Fc \right\}.$$

Further inspection shows that the contravariant f^* is the identity map, since it maps by the identity on the component in $\mathcal{E}$ and its map on the hom-set is unique up to isomorphism, since $*$ is a terminal element in the category. Thus, we can write:

$$\mathrm{Lan}_i(F)(*) = \mathrm{coeq} \left\{ F(f) : F(c) \to F(c') : f \in \mathrm{arr}(C) \right\}.$$

That is, $\mathrm{Lan}_i(F)(*)$ is the coequalizer of the image of F. This is an alternative definition of the colimit cone under F.

3. We again consider the universal property that $\mathrm{Lan}_i(F)(*)$ must satisfy. Specifically, it is isomorphic to the colimit of the composition,

$$i(\mathcal{C})/* \xrightarrow{U} \mathcal{C} \xrightarrow{F} \mathcal{E}.$$

In our case, $i(\mathcal{C})/*$ is isomorphic to $\mathcal{C}$, since it contains all of the morphisms of $\mathcal{C}$ (all of the new composition triangles will automatically commute by the universal property of the terminal object) and all of the objects (each object in $i(\mathcal{C})/*$ is the unique morphism $c \to *$, which maps by U to c).

So the map U is an isomorphism, and the resulting colimit is consequently isomorphic to $\mathrm{colim}\, F : \mathcal{C} \to \mathcal{E}$, as desired.

■

Exercise 2.2.5 ([Rie14] 1.4.3)**.** If $F : \mathcal{C} \leftrightarrows \mathcal{D} : G$ is an adjunction with unit $\eta : 1 \Rightarrow GF$ and counit $\varepsilon : FG \Rightarrow 1$, then (G, η) is a left Kan extension of the identity functor at $\mathcal{C}$ along F and (F, ε) is a right Kan extension of the identity functor at $\mathcal{D}$ along G. Conversely, if (G, η) is a left Kan extension of the identity along F and F preserves this Kan extension, then $F \dashv G$ with unit η.

Proof. We use the fact that we are given that an adjunction $F \dashv G$ gives an adjunction between $G^* \dashv F^*$[1]. Recall that left and right Kan extensions along a functor, F, define left and right adjoints to the precomposition functor F^*. Thus, if we consider the precomposition adjunction we get from picking the base category to be $\mathcal{D}$, we get the diagram:

$$\mathcal{D}^{\mathcal{C}} \underset{F^*}{\overset{G^*}{\underset{\perp}{\leftrightarrows}}} \mathcal{D}^{\mathcal{D}}$$

And by the uniqueness of adjoints (since the left Kan extension along F would also define a left adjoint to F^*), everything in the image of G^* is a left Kan extension along F. Thus, we have $\mathrm{Lan}_F(\mathrm{Id}_{\mathcal{D}}) = G^*(\mathrm{Id}_{\mathcal{D}}) = G$. Furthermore, if the precomposition adjunction has unit $H : \mathrm{Id}_{\mathcal{D}^{\mathcal{D}}} \Rightarrow F^*G^* = (GF)^*$, then the leg corresponding to $\mathrm{Id}_{\mathcal{D}}$ is the natural transformation, $\eta : \mathrm{Id}_{\mathcal{D}} \Rightarrow (GF)^*(\mathrm{Id}_{\mathcal{D}}) = GF$ that defined the initial

[1] The verification of the relevant triangle identities follows by whiskering the relevant functors and natural transformations on the right of the diagram for the original triangle identities.

adjunction. So (G, η) is the object in the image of $\mathrm{Id}_{\mathcal{D}}$ by G, the left adjoint of F^*, and is thus the Left Kan extension of $\mathrm{Id}_{\mathcal{D}}$ along F. A dual argument gives that (F, ε) is the appropriate right Kan extension.

On the other hand, if $(G, \eta) = \mathrm{Lan}_F(\mathrm{Id}_{\mathcal{D}})$, and F preserves this Kan extension, we have that $(FG, F\eta) = \mathrm{Lan}_F(F)$. To show that this is enough data to give an adjunction between F and G, we derive the counit through the universal property of the left Kan extension, $\mathrm{Lan}_F(F) = FG$. Note that we have the natural transformation $\mathrm{Id}_F : F \Rightarrow \mathrm{Id}_{\mathcal{D}}F$ given by $\varphi_c = \mathrm{Id}_{Fc}$. Thus, the universal property of the left Kan extension gives a (unique) natural transformation, $\varepsilon : \mathrm{Id}_{\mathcal{D}} \Rightarrow FG$ such that $F\eta \circ \varepsilon F = \mathrm{Id}_F$.

It suffices to show that these natural transformations satisfy the triangle identities necessary to form an adjunction. That is, $\varepsilon F \circ F\eta = \mathrm{Id}_F$ and $G\varepsilon \circ G\eta = \mathrm{Id}_G$. The first identity follows immediately from the construction of ε, since ε is the unique natural transformation such that the composition $F\eta \circ \varepsilon F = \mathrm{Id}_F$.

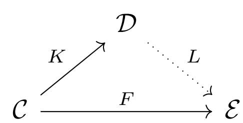
We next verify $\mathrm{Id}_G = G\varepsilon \circ \eta G$. The uniqueness condition of the universal property of the Kan extension G implies this equality is equivalent to $\mathrm{Id}_{GF} \circ \eta = (G\varepsilon \circ \eta G)F \circ \eta$. We verify this latter equation by straightforward algebraic manipulation appealing to the properties of whiskering 1.3.3, the interchange law 1.3.4, and the triangle equation for F:

$$(G\varepsilon \circ \eta G)F \circ \eta = G\varepsilon F \circ \eta GF \circ \eta = G\varepsilon F \circ GF\eta \circ \eta = G(\varepsilon F \circ F\eta) \circ \eta = G(\mathrm{Id}_F) \circ \eta = \mathrm{Id}_{GF} \circ \eta$$

■

2.3 Geometric Realization

Construction 2.3.1 ([Rie14], 1.5.1, Generalized)**.** Let $\mathcal{C}$ be any small category, $\mathcal{D}$ be any locally small category, and let $K : \mathcal{C} \to \mathcal{D}$ be a fully faithful functor. Let $\mathcal{E}$ be any co-complete, locally small category, let and let $F : \mathcal{C} \to \mathcal{E}$ be any contravariant functor. Define $L : \mathcal{D} \to \mathcal{E}$ to be the left Kan extension of F along K.



.

Because $\mathcal{E}$ is assumed to be cocomplete, $\mathcal{C}$ to be small and $\mathcal{D}$ to be locally small, the functor L is defined on objects by the coend 2.2.1

$$Ld := \int^{c \in \mathcal{C}} \mathcal{D}(Kc, d) \cdot Fc = \int^{c \in \mathcal{C}} \mathcal{D}(Kc, d) \cdot Fc \cong \mathrm{coeq}\left(\coprod_{f : c \to c'} \mathcal{D}(Kc', d) \cdot Fc \rightrightarrows \coprod_{c \in \mathcal{C}} \mathcal{D}(Kc, d) \cdot Fc \right).$$

The functor L on arrows is defined by universal property of these colimits. The uniqueness property from the universal property will imply that L is functorial. Since K is fully faithful, $LK \cong F$ [Rie14, 1.4.5]. Because L is defined by a colimit and colimits commute with each other, L preserves colimits. Now since $\mathcal{D}$ is small and L is co-continuous, the Adjoint Functor Theorem ([Rie17, 4.6.1]) implies L admits a right adjoint $R : \mathcal{E} \to \mathcal{D}$.

Now let $\mathcal{D} = \mathbf{Set}^{\mathcal{C}^{\mathrm{op}}}$, and K defined by $K(c) = \mathcal{C}(-, c)$. Then from the Yoneda lemma and $L \dashv R$ we get,

$$(Re)(c) \cong \mathbf{Set}^{\mathcal{C}^{\mathrm{op}}}(\mathcal{C}(-, c), Re) \cong \mathcal{E}(LK(c), e) \cong \mathcal{E}(F(c), e).$$

With the help of this isomorphism we now define $R : \mathcal{E} \to \mathbf{Set}^{\mathcal{C}^{\mathrm{op}}}$ on objects as $(Re)(c) = \mathcal{E}(F(c), e)$, and on morphisms by

$$(Re)(f : c \to c') = \mathcal{E}(F(f), e) : (Re)(c') \to (Re)(c).$$

Then, $(R(g : e \to e')) : Re \Rightarrow Re'$ is defined by $(R(g : e \to e'))_c = \mathcal{E}(Fc, g)$. This is indeed a natural transformation, since

$$\mathcal{E}(Fc, g) \circ \mathcal{E}(Ff, e) = \mathcal{E}(Ff, e') \circ \mathcal{E}(Fc', g).$$

This makes R into a functor.

Definition 2.3.2 ([Rie14], 1.5.3)**.** In the above construction let $\mathcal{C} = \mathbb{\Delta}$, $\mathcal{D} = \mathbf{Set}^{\mathbb{\Delta}^{\text{op}}}$ and $K = \Delta^{\bullet}$. Let $\mathcal{E} = \mathbf{Top}$, then there is a natural functor $F : \mathbb{\Delta} \to \mathbf{Top}$ such that Fn is the standard topological $n-$ simplex,

$$Fn := \left\{ (x_0, \dots, x_n) \in \mathbb{R}^{n+1} \middle| x_i \geq 0, \sum_i x_i = 1 \right\}$$

with the subspace topology. Then the functor $L : \mathbf{sSet} \to \mathbf{Top}$ is called the **geometric realization functor**. From the above prescription $|\cdot| := L : \mathbf{sSet} \to \mathbf{Top}$ is defined at a simplicial set X by

$$|X| = \int^n X_n \times \Delta^n = \text{colim} \left(\coprod_{f:[n]\to[m]} X_m \times \Delta^n \rightrightarrows \coprod_{[n]} X_n \times \Delta^n \right)$$

and the associated right adjoint $R : \mathbf{Top} \to \mathbf{sSet}$ is called the **total singular complex functor.**

Exercise 2.3.3 ([Rie14], 1.5.4)**.** Prove that geometric realization is left adjoint to the singular complex functor by demonstrating this fact for any adjunction arising from the construction of [Rie14, 1.5.1]

Proof. This is a special case of the construction 2.3.1. ∎

2.4 Day Convolution

We encountered the *Day Convolution* as an important example of a Kan extension (used in the construction of a symmetric monoidal structure on the category of spectra), which motivated the problem following this section.

Definition 2.4.1. (n-lab) Let $\mathcal{C}$ be a complete, co-complete, closed, small V-enriched monoidal category (we will mostly deal with $V = \mathbf{Vect}_k$, i.e $\mathcal{C}$ is a small linear category) Let F, G be enriched functors $\mathcal{C} \to V$, and define the *external tensor product*

$$\overline{\otimes} \colon [\mathcal{C}, V] \times [\mathcal{C}, V] \to [\mathcal{C} \times \mathcal{C}, V]$$

by

$$(X \overline{\otimes} Y)(c_1, c_2) := X(c_1) \otimes_V Y(c_2)$$

Then the *Day Convolution* of F and G is defined to be the left Kan extension:

$$\begin{array}{ccc} \mathcal{C} \times \mathcal{C} & \xrightarrow{\;F \overline{\otimes} G\;} & V \\ & {}_{\otimes}\searrow \quad \Downarrow \eta \quad \nearrow_{F \otimes_{\text{Day}} G} & \\ & \mathcal{C} & \end{array}$$

Problem. Let G be a finite group, ω a 3-cocycle, and $\mathcal{C} = \mathbf{Rep}_G^{\omega}$ be the twisted represention category over a field k of characteristic coprime to G. The set of enriched functors $\mathcal{C} \to \mathbf{Vect}_k$ (up to natural isomorphism) has a multiplication given by Day convolution.

- The natural transformation η in the previous diagram is not specified in the n-lab. What is it?
- Let $k = \mathbb{C}, \omega = 1$ and $\iota \colon \mathbf{Rep}_G \to \mathbf{Vect}_{\mathbb{C}}$ be the natural inclusion. What is the tensor product $\iota \otimes_{\text{Day}} \iota$?
- Can we compute the ring structure on the functors $\mathbf{Rep}_G^{\omega} \to \mathbf{Vect}_{\mathbb{C}}$? Is this a "classical" invariant of G if $\omega = 1$?
- What if G is (compact) topological and everything in sight is required to be continuous?

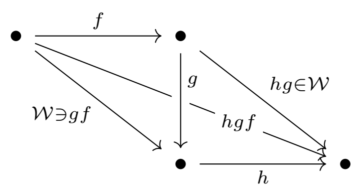
Note: It seems like the coend formula defining the Day convolution should simplify significantly, perhaps just to the colimit over the generating simple objects in any semisimple category.

Chapter 3

Derived Functors via Deformations

3.1 Weak Equivalences and Homotopy Categories

Definition 3.1.1 ([Rie14], 2.1.1)**.** A **homotopical category** is a category $\mathcal{M}$ equipped with a wide (lluf) subcategory $\mathcal{W}$ such that for any composable triple of



$\Rightarrow f, g, h, hgf \in \mathcal{W}$

if hg and gf are in $\mathcal{W}$ so are f, g, h, hgf.
The arrows in $\mathcal{W}$ are called **weak equivalences**; above condition is called **2-of-6 property**.

Definition 3.1.2 (n-lab)**.** Let $\mathcal{C}$ be a category and $W \subset \mathrm{mor}(\mathcal{C})$. A **localization** of $\mathcal{C}$ by W is a category $\mathcal{C}[W^{-1}]$ and a functor $Q : \mathcal{C} \to \mathcal{C}[W^{-1}]$ such that:

1. for all $w \in W$, $Q(W)$ is an isomorphism;
2. for any category $\mathcal{E}$ and any functor $F : \mathcal{C} \to \mathcal{E}$ such that $F(w)$ is an isomorphism for all $w \in W$, there exists a functor $F_W : C[W^{-1}] \to A$ and a natural isomrphism $F \cong F_W \circ Q$;
3. the map between functor categories

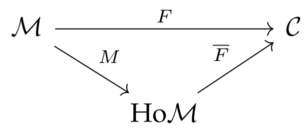
$$Q_* = - \circ Q : \mathcal{A}^{\mathcal{C}[W^{-1}]} \to \mathcal{A}^{\mathcal{C}}$$

is full and faithful for every category A.

Note that: if $\mathcal{C}[W^{-1}]$ exists then it is unique up to equivalence.

Notation 3.1.3. Riehl typically uses a γ or δ for the localization functor. We will instead use capital Roman letters (with the same letter as their domain) instead of lowercase Greek. This way our notation is consistent in the sense that script letters are typically categories, capital Roman letters are typically functors, and lowercase Greek letters are almost always natural transformations. Hopefully, this helps avoid compositions that don't typecheck.
Definition 3.1.4 ([Rie14], 2.1.6)**.** The **homotopy category** Ho$\mathcal{M}$ of a homotopical category $(\mathcal{M}, \mathcal{W})$ is the formal localization of $\mathcal{M}$ at the subcategory $\mathcal{W}$. That is given any functor $F : \mathcal{M} \to \mathcal{C}$ that maps weak equivalences to isomorphisms, which we call **homotopical**, there is a functor $\overline{F} : \mathrm{Ho}\mathcal{M} \to \mathcal{C}$ such that

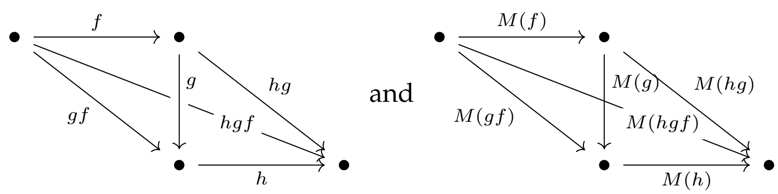
following diagram commutes.



The functor $\overline{F}$ is unique up to natural isomorphism.

Lemma ([Rie14], 2.1.10)**.** *Let $\mathcal{M}$ be a category equipped with any collection of arrows $\mathcal{W}$. If the localization Ho$\mathcal{M} := \mathcal{M}[\mathcal{W}^{-1}]$ is saturated, then $\mathcal{W}$ satisfies the 2-of-6 property.*

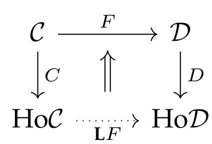
Proof. Suppose we have morphisms f, g, h of $\mathcal{M}$ with $\bullet \xrightarrow{f} \bullet \xrightarrow{g} \bullet \xrightarrow{h} \bullet$ and $gf, hg \in \mathcal{W}$. Let M be the localization functor. Then the diagrams



are commutative, and $M(fg), M(gh)$ are isomorphisms. Since the isomorphisms in any category satisfy 2-of-6 ([Rie14], 2.1.4), the morphisms $M(f), M(g), M(h)$ and $M(hgf)$ are isomorphisms. By the definition of saturation $f, g, h, hgf \in \mathcal{W}$. So $\mathcal{W}$ satisfies 2-of-6, as desired. ■

3.2 Derived functors

Definition 3.2.1 ([Rie14], 2.1.17)**.** A **total left derived functor L**F of a functor F between homotopical categories $\mathcal{C}$ and $\mathcal{D}$ is a right Kan extension Ran$_C DF$



where C and D are the localization functors for $\mathcal{C}$ and $\mathcal{D}$.
Definition 3.2.2 ([Rie14], 2.1.19)**.** A **left derived functor** of $F : \mathcal{C} \to \mathcal{D}$ is a homotopical functor $\mathbb{L}F : \mathcal{C} \to \mathcal{D}$ equipped with a natural transformation $\lambda : \mathbb{L}F \Rightarrow F$ such that $D \cdot \lambda : D \circ \mathbb{L}F \Rightarrow D \circ F$ is a total left derived functor of F.
Definition 3.2.3 ([Rie14], 2.2.1)**.** A **left deformation** on a homotopical category $\mathcal{C}$ consists of an endofunctor Q together with a natural weak equivalence $q : Q \overset{\sim}{\Rightarrow} 1$.
Definition 3.2.4. A **left deformation** for a functor $F : \mathcal{C} \to \mathcal{D}$ between homotopical categories consists of a left deformation for $\mathcal{C}$ such that F is homotopical on an associated subcategory of cofibrant objects. When F admits a left deformation, we say that F is left deformable.

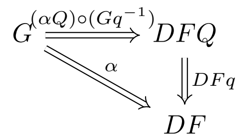
Theorem 3.2.5 ([Dwy05], 41.2-5)**.** *If $F : \mathcal{C} \to \mathcal{D}$ has a left deformation, $q : Q \overset{\sim}{\Rightarrow} 1$, then $\mathbb{L}F = FQ$ is a left derived functor of F.*

Proof. Let C and D be as in 3.2.1. To show that FQ is a left derived functor, it is sufficient to show that DFQ is a *total* left derived functor with the natural transformation $DFq : DFQ \Rightarrow DF$.

Note that it is reasonable to call $DFQ : \mathcal{C} \to$ Ho$\mathcal{D}$ a total left derived functor because of the following reasoning: Since F has a left deformation, Q, it is homotopical on M_Q, since Q is necessarily homotopical, FQ is homotopical, so is DFQ. Then by the 2-Categorical Universal Property of the localization functor C, homotopical functors and natural transformations between them, $DFq : DFQ \Rightarrow DF$ are in bijection with the functors and natural transformations induced by C, $\overline{DFq} : \overline{DFQ} \Rightarrow \overline{DF}$. Thus, we can prove the desired condition for the functors from $\mathcal{C}$, since the "actual" condition will be met if and only if the one with $\mathcal{C}$ is a domain is.

Thus we need to show that for any homotopical functor, $G : C \to \mathrm{Ho}\mathcal{D}$, and a natural transformation, $\alpha : G \Rightarrow DF$, there exists a unique natural transformation, $\eta : G \Rightarrow DFQ$. We note that again by the correspondence from the last paragraph, G must be homotopical, since it is also induces a functor on $\mathrm{Ho}\mathcal{C}$, which automatically preserves isomorphisms, and is thus homotopical to $\mathrm{Ho}\mathcal{D}$, and the bijection from C is on homotopical functors. Next we note that the whiskered natural weak equivalence $Gq : GQ \Rightarrow G$ is actually an isomorphism, since G is homotopical with target $\mathrm{Ho}\mathcal{D}$ (where all weak equivalences are isomorphisms).

We suggest the composition of two-cells, $(\alpha Q)\circ(Gq^{-1})$. Since Gq was a natural isomorphism this is a well-defined natural transformation. We now need to check that this natural transformation fits the 2-Categorical condition of the Kan extension. I.e. the following diagram of natural transformations commutes:



This follows from the interchange law, since the natural transformations are occurring separately:

$$(DFq) \circ (\alpha Q) \circ (Gq)^{-1} = (DFq) \circ (DFq^{-1}) \circ \alpha = \alpha.$$

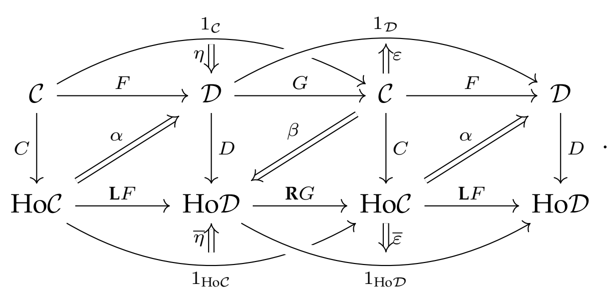
Which is the desired property. ∎

Exercise 3.2.6 ([Rie14], 2.2.15)**.** Suppose $F \dashv G$ is an adjunction between homotopical categories and suppose also that F has a total left derived functor $(\mathbf{L}F, \alpha)$, G has a total right derived functor $(\mathbf{R}G, \beta)$ and both derived functors are absolute Kan extensions. Show that $\mathbf{L}F \dashv \mathbf{R}G$. That is, show the total derived functors form an adjunction between the homotopy categories, regardless of how these functors may have been constructed.

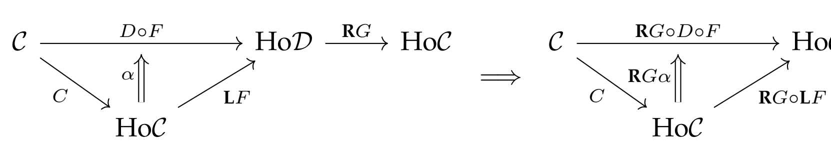
Proof. **Step I:** Finding unit/ co-unit for $\mathbf{L}F \dashv \mathbf{R}G$.
Let $\eta : F \circ G \Rightarrow 1_{\mathcal{D}}$ and $\varepsilon : 1_{\mathcal{C}} \Rightarrow G \circ F$ be units and co-units of $F \dashv G$ adjunction. Then we have adjunction equations

$$(\varepsilon F)(F\eta) = 1_F \quad \text{and} \quad (G\varepsilon)(\eta G) = 1_G.$$

The following master diagram is helpful in tracking all functors and natural transformations:



Using that $\mathbf{R}G$ is an absolute Kan extension we get,



to be a Kan extension. Now consider

$$\gamma = (\beta F)(C\eta) : 1_{\mathrm{Ho}\mathcal{C}} \circ C = C \xRightarrow{C\eta} C \circ G \circ F \xRightarrow{\beta F} \mathbf{R}G \circ D \circ F$$

Then from the universal property of the Kan extension, there is a (unique) natural transformation $\overline{\eta} : 1_{\text{Ho}\mathcal{C}} \Rightarrow \mathbf{R}G \circ \mathbf{L}F$ such that,

$$(\mathbf{R}G\alpha)(\overline{\eta}C) = (\beta F)(C\eta).$$

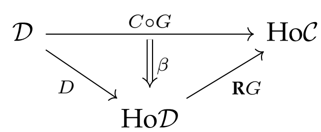
Similarly we get $\overline{\varepsilon} : \mathbf{L}F \circ \mathbf{R}G \Rightarrow 1_{\text{Ho}\mathcal{D}}$ such that,

$$(\overline{\varepsilon}D)(\mathbf{L}F\beta) = (D\varepsilon)(\alpha G)$$

Step II: Checking unit/ co-unit equations for $\mathbf{L}F \dashv \mathbf{R}G$.
We need to show that

$$(\mathbf{R}G\,\overline{\varepsilon})(\overline{\eta}\,\mathbf{R}G) = 1_{\mathbf{R}G} \quad \text{and} \quad (\overline{\varepsilon}\,\mathbf{L}F)(\mathbf{L}F\,\overline{\eta}) = 1_{\mathbf{L}F}$$

For the first equation we will use the following Kan extension,



We claim that

$$[(\mathbf{R}G\,\overline{\varepsilon})(\overline{\eta}\,\mathbf{R}G)D](\beta) = \beta$$

Then from the universal property of Kan extensions, we get $(\mathbf{R}G\,\overline{\varepsilon})(\overline{\eta}\,\mathbf{R}G) = 1_{\mathbf{R}G}$. The claim is true from following chain of equalities ([Mal07]),

$$\begin{aligned}
\big[\big((\mathbf{R}G\,\overline{\varepsilon})(\overline{\eta}\,\mathbf{R}G)\big)D\big](\beta) &= \big(\mathbf{R}G\,\overline{\varepsilon}\,D\big)\big(\overline{\eta}\,(\mathbf{R}G \circ D)\big)(\beta) && (1)\\
&= \big(\mathbf{R}G\,\overline{\varepsilon}\,D\big)\big((\mathbf{R}G \circ \mathbf{L}F)\beta\big)\big(\overline{\eta}\,(C \circ G)\big) && (2)\\
&= \big[\mathbf{R}G\big((\overline{\varepsilon}\,D)(\mathbf{L}F\,\beta)\big)\big]\big(\overline{\eta}\,(C \circ G)\big) && (3)\\
&= \big[\mathbf{R}G\big((D\,\varepsilon)(\alpha\,G)\big)\big]\big(\overline{\eta}\,(C \circ G)\big) && (4)\\
&= \big((\mathbf{R}G \circ D)\,\varepsilon\big)\big(\mathbf{R}G\,\alpha\,G\big)\big(\overline{\eta}\,(C \circ G)\big) && (5)\\
&= \big((\mathbf{R}G \circ D)\,\varepsilon\big)\big[\big((\mathbf{R}G\,\alpha)(\overline{\eta}\,C)\big)\,G\big] && (6)\\
&= \big((\mathbf{R}G \circ D)\,\varepsilon\big)\big[\big((\beta\,F)(C\,\eta)\big)\,G\big] && (7)\\
&= \big((\mathbf{R}G \circ D)\,\varepsilon\big)\big(\beta\,(F \circ G)\big)\big(C\,\eta\,G\big) && (8)\\
&= \big(\beta\big)\big((C \circ G)\,\varepsilon\big)\big(C\,\eta\,G\big) && (9)\\
&= \big(\beta\big)\big[C\,\big((G\,\varepsilon)(\eta\,G)\big)\big] && (10)\\
&= \beta && (11)
\end{aligned}$$

Here equalities 1, 3, 5, 6, 8, and 10 are appropriate whiskering properties listed at proposition 1.3.3. Equalities 2 and 9 are from proposition 1.3.4. Equalities 4 and 7 are defining equations of $\overline{\varepsilon}$ and $\overline{\eta}$ respectively. And the last equality is unit-co-unit equation.
Thus we get $(\mathbf{R}G\,\overline{\varepsilon})(\overline{\eta}\,\mathbf{R}G) = 1_{\mathbf{R}G}$. Similarly we get $(\overline{\varepsilon}\,\mathbf{L}F)(\mathbf{L}F\,\overline{\eta}) = 1_{\mathbf{L}F}$. Hence $\mathbf{L}F \dashv \mathbf{R}G$. ■

Chapter 4

Basic Concepts of Enriched Category Theory

4.1 Enriched Categories

Definition 4.1.1 ([ML13]). A **Monoidal Category** $(\mathcal{V}, \times, *)$ is a category, $\mathcal{V}$ and a bifunctor $- \times - : \mathcal{V} \times \mathcal{V} \to \mathcal{V}$ called the **monoidal product**, and $* \in \mathcal{V}$, the **unit object** such that the following properties hold:

1. There exist natural isomorphisms between functors:

$$a \times (b \times c) \cong_\alpha (a \times b) \times c \qquad * \times a \cong_\lambda a \cong_\rho a \times *.$$

2. And the associater satisfies the following compatibility condition:

$$\begin{array}{ccccc} a \times (b \times (c \times d)) & \xrightarrow{\alpha_{cd}} & (a \times b) \times (c \times d) & \xrightarrow{\alpha_{ab}} & ((a \times b) \times c) \times d \\ \downarrow{\scriptstyle \mathrm{Id}_a \cdot \alpha} & & & & \downarrow{\scriptstyle \alpha \cdot \mathrm{Id}_d} \\ a \times ((b \times c) \times d) & & \xrightarrow{\alpha_{bc}} & & (a \times (b \times c)) \times d \end{array}$$

3. While the left and right unitors satisfy the following two conditions:
   This diagram commutes exactly,

$$\begin{array}{ccc} a \times (* \times b) & \xrightarrow{\alpha} & (a \times *) \times b \\ & \searrow^{\mathrm{Id}_a \cdot \lambda} & \downarrow{\scriptstyle \rho \cdot \mathrm{Id}_b} \\ & & a \times b \end{array}$$

   and the unit object components of ρ and λ are the same:

$$\lambda_* = \rho_* : * \times * \to *.$$

Definition 4.1.2 ([ML13]). A **braiding**, γ, is a natural isomorphism,

$$a \times b \cong_\gamma b \times a$$

which is compatible with the left and right unitors:

$$\begin{array}{ccc} a \times * & \xrightarrow{\gamma} & * \times a \\ & \searrow^{\rho} & \downarrow{\scriptstyle \lambda} \\ & & a \end{array}$$

and is compatible with the associater on the following hexagonal conditions:

$$\begin{array}{ccc} (a\times b)\times c & \xrightarrow{\gamma_{ab}} & c\times(a\times b)\\ \Big\downarrow{\scriptstyle \alpha^{-1}} & & \Big\downarrow{\scriptstyle \alpha}\\ a\times(b\times c) & & (c\times a)\times b\\ \Big\downarrow{\scriptstyle \mathrm{Id}_a\gamma} & & \Big\downarrow{\scriptstyle \gamma\mathrm{Id}_b}\\ a\times(c\times b) & \xrightarrow{\alpha} & (a\times c)\times b \end{array}$$

There is another hexagon, but for the case of a symmetric monoidal category, it is implied by this hexagon.
Definition 4.1.3 ([ML13]). A monoidal category is **symmetric** if it has a braiding γ such that

$$\begin{array}{ccc} a\times b & \xrightarrow{\gamma} & b\times a\\ & {\scriptstyle \mathrm{Id}_{a\times b}}\searrow & \Big\downarrow{\scriptstyle \gamma}\\ & & a\times b \end{array}$$

Definition 4.1.4 ([Rie14], 3.3.1). For a symmetric monoidal category $(\mathcal{V},\times,*)$, a $\mathcal{V}$-**category** $\underline{\mathcal{D}}$ consists of

1. a collection of objects $x,y,z\in\underline{\mathcal{D}}$,
2. for each pair $x,y\in\underline{\mathcal{D}}$, a **hom-object** $\mathcal{D}(x,y)\in\mathcal{V}$,
3. for each $x\in\underline{\mathcal{D}}$, a morphism $\mathrm{Id}_x : *\to\underline{\mathcal{D}}(x,x)$ in $\mathcal{V}$,
4. and for each triple $x,y,z\in\underline{\mathcal{D}}$, a morphism $\circ : \underline{\mathcal{D}}(y,z)\times\underline{\mathcal{D}}(x,y)\to\underline{\mathcal{D}}(x,z)$ in $\mathcal{V}$ such that following diagrams commute for all $x,y,z,w\in\underline{\mathcal{D}}$:

$$\begin{array}{ccc} \underline{\mathcal{D}}(z,w)\times\underline{\mathcal{D}}(y,z)\times\underline{\mathcal{D}}(x,y) & \xrightarrow{1\times\circ} & \underline{\mathcal{D}}(z,w)\times\underline{\mathcal{D}}(x,z)\\ \Big\downarrow{\scriptstyle \circ\times 1} & & \Big\downarrow{\scriptstyle \circ}\\ \underline{\mathcal{D}}(y,w)\times\underline{\mathcal{D}}(x,y) & \xrightarrow{\qquad\circ\qquad} & \underline{\mathcal{D}}(x,w) \end{array}$$

$$\begin{array}{ccc} \underline{\mathcal{D}}(x,y)\times * & \xrightarrow{1\times\mathrm{Id}_x} & \underline{\mathcal{D}}(x,y)\times\underline{\mathcal{D}}(x,x)\\ & {\scriptstyle\cong}\searrow & \Big\downarrow{\scriptstyle \circ}\\ & & \underline{\mathcal{D}}(x,y) \end{array} \qquad \begin{array}{ccc} \underline{\mathcal{D}}(y,y)\times\underline{\mathcal{D}}(x,y) & \xleftarrow{\mathrm{Id}_y\times 1} & *\times\underline{\mathcal{D}}(x,y)\\ \Big\downarrow{\scriptstyle \circ} & \swarrow{\scriptstyle\cong} & \\ \underline{\mathcal{D}}(x,y). & & \end{array}$$

Here the indicated isomorphisms are the maps defined in 4.1.1.

Definition 4.1.5 ([Rie14], 3.3.6). A (symmetric) monoidal category $\mathcal{V}$ is called a **closed monoidal category** when each functor $-\times v : \mathcal{V}\to\mathcal{V}$ admits a right adjoint $\underline{\mathcal{V}}(v,-)$, the right adjoints in this family of parametrized adjunctions assemble in a unique way into a bifunctor

$$\mathcal{V}(-,-) : \mathcal{V}^{op}\times\mathcal{V}\to\mathcal{V}$$

such that there exists isomorphisms

$$\mathcal{V}(u\times v,w)\cong\mathcal{V}(u,\underline{\mathcal{V}}(v,w)) \qquad \forall u,v,w\in\mathcal{V}$$

natural in all three variables.

4.2 Underlying categories of enriched categories

Definition 4.2.1 ([Rie14], 3.4.4, 3.4.5)**.** The **underlying category** $\mathcal{C}_0$ of a $\mathcal{V}$-category $\underline{\mathcal{C}}$ has the same objects and has hom-sets

$$C_0(x,y) := \mathcal{V}(*, \underline{\mathcal{C}}(x,y)).$$

We define the identities $\mathrm{Id}_x \in C_0(x,x)$ to be the specified morphisms $\mathrm{Id}_x \in \mathcal{V}(*, \underline{\mathcal{C}}(x,x))$. Composition is defined hom-wise by the following dashed arrow in **Set**:

$$\begin{array}{ccccc} C_0(y,z) \times C_0(x,y) & & \dashrightarrow & & C_0(x,z) \\ \| & & & & \| \\ \mathcal{V}(*, \underline{\mathcal{C}}(x,y)) \times \mathcal{V}(*, \underline{\mathcal{C}}(x,y)) & \longrightarrow & \mathcal{V}(*, \underline{\mathcal{C}}(y,z) \times \underline{\mathcal{C}}(x,y)) & \longrightarrow & \mathcal{V}(*, \underline{\mathcal{C}}(x,y)) \end{array}$$

The first arrow in the bottom row is the natural morphism (from [Rie14], 3.4.4) $\mathcal{V}(*,u) \times \mathcal{V}(*,w)$ defined by precomposing $f \times g$ with the natural isomorphism $* \cong * \times *$. The second is $\mathcal{V}(*,-)$ applied to the composition morphism for $\underline{\mathcal{C}}$.

Exercise 4.2.2 ([Rie14], 3.4.13)**.** Let $\underline{\mathcal{D}}$ be a $\mathcal{V}-$category and $\; * \xrightarrow{g} \underline{\mathcal{D}}(y,z) \;$ be an arrow in $\mathcal{D}_0$. Using g define for any $x \in \underline{\mathcal{D}}$

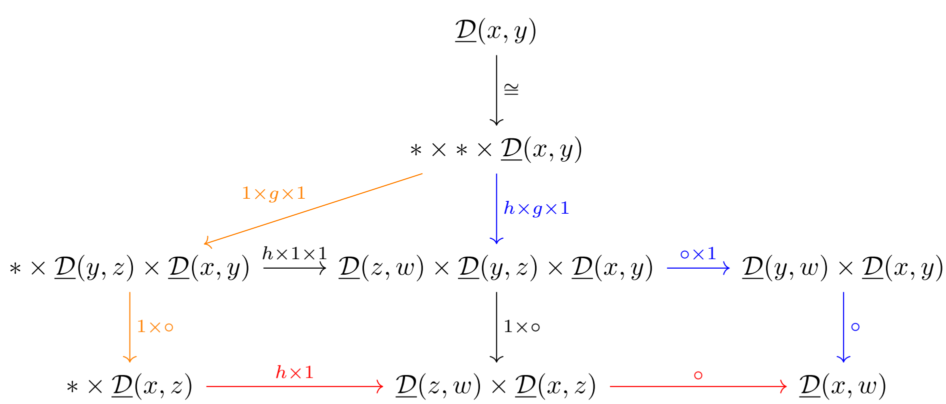
$$g_* : \underline{\mathcal{D}}(x,y) \cong * \times \underline{\mathcal{D}}(x,y) \xrightarrow{g \times 1} \underline{\mathcal{D}}(y,z) \times \underline{\mathcal{D}}(x,y) \xrightarrow{\circ} \underline{\mathcal{D}}(x,z).$$

This construction defines an (unenriched) representable functor

$$\underline{\mathcal{D}}(x,-) : \mathcal{D}_0 \to \mathcal{V}$$

Show the composite of this functor with the underlying set functor $\mathcal{V}(*,-) : \mathcal{V} \to$ **Set** is the representable functor $\mathcal{D}_0(x,-) : \mathcal{D}_0 \to$ **Set** for the underlying category $\mathcal{D}_0$.

Proof. Define $\underline{\mathcal{D}}(x,-) : \mathcal{D}_0 \to \mathcal{V}$ by $\underline{\mathcal{D}}(x,-)(y) = \underline{\mathcal{D}}(x,y)$ on objects and $\mathcal{D}(x,-)(g : * \to \underline{\mathcal{D}}(y,z)) = g_* : \underline{\mathcal{D}}(x,y) \to \underline{\mathcal{D}}(x,z)$. For simplicity we will write $g_* = (- \circ -)(g \times 1) =: g \circ 1$. Let $h : * \to \underline{\mathcal{D}}(z,w)$ be another arrow in $\mathcal{D}_0$ then, $hg = (- \circ -)(h \times 1) = h \circ g$. Functoriality follows from the associativity in the enriched category $\underline{\mathcal{D}}$,



Here top-left triangle and bottom left rectangle commutes since the operations are in different co-ordinates. Bottom right rectangle is associativity in $\underline{\mathcal{D}}$. Orange (left composition) and red (bottom composition) arrows are g_* and h_* respectively, together they are $h_* g_*$. Whereas the blue (right composition) arrows are $(hg)_*$. Thus $(hg)_* = h_* g_*$. Thus $\underline{\mathcal{D}}(x,-)$ is a functor.
Now let $F : \mathcal{D}_0 \to$ **Set** be $F = \mathcal{V}(*,-) \circ \underline{\mathcal{D}}(x,-)$. Then on objects,

$$F(y) = \mathcal{V}(*, \underline{\mathcal{D}}(x,y)) = \mathcal{D}_0(x,y).$$

Let $g : y \to z$ be an arrow in $\mathcal{D}_0$, that is $g : * \to \underline{\mathcal{D}}(y,z)$. Let $h : x \to y \in \mathcal{D}_0(x,y)$, that is $h : * \to \underline{\mathcal{D}}(x,y)$, then we have

$$\begin{array}{ccccc} * \cong * \times * & & & & \\ \downarrow{\scriptstyle 1\times h} & \searrow^{g\times h} & & & \\ * \times \underline{\mathcal{D}}(x,y) & \xrightarrow{g\times 1} & \underline{\mathcal{D}}(y,z) \times \underline{\mathcal{D}}(x,y) & \xrightarrow{\circ} & \underline{\mathcal{D}}(x,z) \end{array}.$$

The diagram commutes since operations take place in different co-ordinates. Blue (diagonal) arrow with composition arrow is post-compositon by g, and red (left bottom) arrows with composition is postcomposition by g_*, thus

$$\mathcal{D}_0(x,g)(h) = \mathcal{V}(*, g_*)(h) = \mathcal{V}(*, \underline{\mathcal{D}}(x,g))(h) = F(g)(h)$$

Thus $F = \mathcal{D}_0(x,-)$. ■

Exercise 4.2.3 ([Rie14], 3.5.3)**.** Extend Definition 4.2.1 (of Underlying Category) to define the underlying functor of a $\mathcal{V}$-Functor and show that your definition is functorial. i.e. it defines a functor

$$(-)_0 : \mathcal{V}\text{-}\mathbf{Cat} \to \mathbf{Cat}.$$

Proof. Suppose that $\underline{F} \colon \underline{\mathcal{C}} \to \underline{\mathcal{D}}$ is a $\mathcal{V}$-Functor. Then, we define the underlying functor, $(\underline{F})_0 = F \colon \mathcal{C} \to \mathcal{D}$ to be the same on objects as $\underline{F}$. We define F on morphisms by taking $f \in C(x,y) = \mathcal{V}(*, \underline{\mathcal{C}}(x,y))$ to $\mathcal{V}(*, F_{x,y})(f)$. It remains to show that F is a functor. Suppose that $f \in C(x,y), g \in C(y,z)$ so that $gf \in C(x,z)$. Then $F(gf) = F(g)F(f)$ by applying the functor $\mathcal{V}(*,-)$ to the diagram defining a $\mathcal{V}$-functor:

$$\begin{array}{ccc} \underline{\mathcal{C}}(y,z) \times \underline{\mathcal{C}}(x,y) & \xrightarrow{\quad\circ\quad} & \underline{\mathcal{C}}(x,z) \\ \downarrow{\scriptstyle F_{y,z}\times F_{x,y}} & & \downarrow{\scriptstyle F_{x,z}} \\ \underline{\mathcal{D}}(Fy,Fz) \times \underline{\mathcal{D}}(Fx,Fy) & \xrightarrow{\quad\circ\quad} & \underline{\mathcal{D}}(Fx,Fz) \end{array}$$

$$\Downarrow \mathcal{V}(*,-)$$

$$\begin{array}{ccc} \mathcal{V}(*, \underline{\mathcal{C}}(y,z) \times \underline{\mathcal{C}}(x,y)) & \xrightarrow{\quad\circ\quad} & \mathcal{V}(*, \underline{\mathcal{C}}(x,z)) =: C(x,z) \\ \downarrow{\scriptstyle \mathcal{V}(*, F_{y,z}\times F_{x,y})} & & \downarrow{\scriptstyle \mathcal{V}(*, F_{x,z})} \\ \mathcal{V}(*, \underline{\mathcal{D}}(Fy,Fz) \times \underline{\mathcal{D}}(Fx,Fy)) & \xrightarrow{\circ} & \mathcal{V}(*, \underline{\mathcal{D}}(Fx,Fz)) =: D(Fx,Fz) \end{array}$$

By functoriality of the hom-functor, the latter square also commutes.

The top path applied to $(g \times f) \in \mathcal{V}(*, \underline{\mathcal{C}}(y,z) \times \underline{\mathcal{C}}(x,y))$ is the definition of $F(gf) := \mathcal{V}(*, F_{x,z})(gf)$ while the bottom path gives:

$$\mathcal{V}(*, F_{y,z})(g) \circ \mathcal{V}(*, F_{x,y})(f) =: F(g)F(f).$$

So the commutativity gives that F preserves arrow composition.

The identity condition is similarly satisfied by applying $\mathcal{V}(*,-)$ to the identity triangle in the definition of a $\mathcal{V}$-functor. Thus, we have that F is a functor.

Functoriality of the map $(-)_0 \colon \mathcal{V}\text{-}\mathbf{Cat} \to \mathbf{Cat}$ follows from functoriality of $\mathcal{V}(*,-)$. If $F \colon \underline{\mathcal{C}} \to \underline{\mathcal{D}}$ and $G \colon \underline{\mathcal{D}} \to \underline{\mathcal{E}}$, then their composition is defined pointwise as $(GF)_{x,y} = G_{Fx,Fy}F_{x,y}$. Thus, we get that $\mathcal{V}(*, GF)$ is defined on hom-sets as $\mathcal{V}(*, G_{Fx,Fy} \circ F_{x,y}) = \mathcal{V}(*, G_{Fx,Fy}) \circ \mathcal{V}(*, F_{x,y})$ by the functoriality of $\mathcal{V}(*,-)$. Thus, the underlying functor of GF is the same on morphisms and objects as the composition of the underlying functors F and G, so the underlying functor map is functorial. ■

Proposition 4.2.4 ([Rie14], 3.5.12)**.** *The following are equivalent:*

(i) *$x, y \in \underline{\mathcal{C}}$ are isomorphic as objects of $\mathcal{C}$.*

(ii) *the representable functors $\mathcal{C}(x,-), \mathcal{C}(y,-) : \mathcal{C} \rightrightarrows$ **Set** are naturally isomorphic*

(iii) *the unenriched representable functors $\underline{\mathcal{C}}(x,-), \underline{\mathcal{C}}(y,-) \colon \mathcal{C} \rightrightarrows \mathcal{V}$ are naturally isomorphic*

(iv) *the representable $\mathcal{V}$-functors $\underline{\mathcal{C}}(x,-), \underline{\mathcal{C}}(y,-) \colon \mathcal{C} \rightrightarrows \underline{\mathcal{V}}$ are $\mathcal{V}$-isomorphic*

Exercise 4.2.5 ([Rie14], 3.5.12)**.** Prove $(i) \Rightarrow (iv)$ above.

Proof. First note that $\mathcal{C} := (\underline{\mathcal{C}})_0$. Now let $f \in \mathcal{C}(x,y) := \mathcal{V}(*, \underline{\mathcal{C}}(x,y))$ where f is an isomorphism. Thus $f : * \to \underline{\mathcal{C}}(x,y)$, let $g : * \to \underline{\mathcal{C}}(y,x)$ be its inverse. We need to define a $\mathcal{V}-$natural transformation $\alpha : \underline{\mathcal{C}}(x,-) \Rightarrow \underline{\mathcal{C}}(y,-)$. For arbitrary but fixed $u \in \underline{\mathcal{C}}$ let α_u be following composition,

$$\alpha_u : \underline{\mathcal{C}}(x,u) \cong \underline{\mathcal{C}}(x,u) \times * \xrightarrow{1\times g} \underline{\mathcal{C}}(x,u) \times \underline{\mathcal{C}}(y,x) \xrightarrow{\circ} \underline{\mathcal{C}}(y,u) \ .$$

Similarly let β_u be following composition,

$$\beta_u : \underline{\mathcal{C}}(y,u) \cong \underline{\mathcal{C}}(y,u) \times * \xrightarrow{1\times f} \underline{\mathcal{C}}(y,u) \times \underline{\mathcal{C}}(x,y) \xrightarrow{\circ} \underline{\mathcal{C}}(x,u) \ .$$

Now consider following diagram,

$$\begin{array}{ccccccc}
\underline{\mathcal{C}}(x,u) \times * \times * & \xrightarrow{1\times g\times 1} & \underline{\mathcal{C}}(x,u) \times \underline{\mathcal{C}}(y,x) \times * & \xrightarrow{1\times 1\times f} & \underline{\mathcal{C}}(x,u) \times \underline{\mathcal{C}}(y,x) \times \underline{\mathcal{C}}(x,y) & \xrightarrow{1\times\circ} & \underline{\mathcal{C}}(x,u) \times \underline{\mathcal{C}}(x,x) \\
 & & \downarrow{\scriptstyle \circ\times 1} & & \downarrow{\scriptstyle \circ\times 1} & & \downarrow{\scriptstyle \circ} \\
 & & \underline{\mathcal{C}}(y,u) \times * & \xrightarrow{1\times f} & \underline{\mathcal{C}}(y,u) \times \underline{\mathcal{C}}(x,y) & \xrightarrow{\circ} & \underline{\mathcal{C}}(x,u)
\end{array} \ .$$

The first square commutes since the operations take place independently, the second square is associatiativity square. Here red (left composition) arrows is α_u and blue (bottom composition) arrows is β_u whereas green (top right composition) arrows precomposed with $1 \times g \times 1$ is identity since $g \circ f = 1$. Thus $\beta_u \circ \alpha_u = 1$. Similarly we can show that $\alpha_u \circ \beta_u = 1$. Thus α_u is an isomorphism for each $u \in \underline{\mathcal{C}}$. Now to show that α is an natural transformation we need to show that the following diagram on the left commutes. Since $\mathcal{V}$ is closed monoidal we get that the left hand side diagram commutes if and only if right hand side diagram commutes.

$$\begin{array}{ccc}
\mathcal{C}(u,v) & \longrightarrow & \mathcal{V}(\mathcal{C}(x,u), \mathcal{C}(x,v)) \\
\downarrow & & \downarrow{\scriptstyle (\alpha_v)_*} \\
\mathcal{V}(\mathcal{C}(y,u), \mathcal{C}(y,v)) & \xrightarrow{\alpha_u^*} & \mathcal{V}(\mathcal{C}(x,u), \mathcal{C}(y,v))
\end{array}
\iff
\begin{array}{ccc}
\mathcal{C}(u,v) \times \mathcal{C}(x,u) & \xrightarrow{\circ} & \mathcal{C}(x,v) \\
\downarrow{\scriptstyle 1\times\alpha_u} & & \downarrow{\scriptstyle \alpha_v} \\
\mathcal{C}(u,v) \times \mathcal{C}(y,u) & \xrightarrow{\circ} & \mathcal{C}(y,v)
\end{array}$$

To see that the right hand side diagram commutes consider following diagram,

$$\begin{array}{ccc}
\underline{\mathcal{C}}(u,v) \times \underline{\mathcal{C}}(x,u) & \xrightarrow{\circ} & \underline{\mathcal{C}}(x,v) \\
{\scriptstyle 1\times\cong}\downarrow & & \downarrow{\scriptstyle \cong} \\
\underline{\mathcal{C}}(u,v) \times \underline{\mathcal{C}}(x,u) \times * & \xrightarrow{\circ\times 1} & \underline{\mathcal{C}}(x,v) \times * \\
{\scriptstyle 1\times 1\times g}\downarrow & & \downarrow{\scriptstyle 1\times g} \\
\underline{\mathcal{C}}(u,v) \times \underline{\mathcal{C}}(x,u) \times \underline{\mathcal{C}}(y,x) & \xrightarrow{\circ\times 1} & \underline{\mathcal{C}}(x,v) \times \underline{\mathcal{C}}(y,x) \\
{\scriptstyle 1\times\circ}\downarrow & & \downarrow{\scriptstyle \circ} \\
\underline{\mathcal{C}}(u,v) \times \underline{\mathcal{C}}(y,u) & \xrightarrow{\circ} & \underline{\mathcal{C}}(y,v)
\end{array} \ .$$

Here top and middle square commutes from the properties of composition. The red (left side composition) arrows are $1 \times \alpha_u$ and blue (left side composition) arrows are α_v. Hence α is an natural isomorphism, which makes the representable $\mathcal{V}$-functors $\underline{\mathcal{C}}(x,-), \underline{\mathcal{C}}(y,-) : \mathcal{C} \rightrightarrows \underline{\mathcal{V}}$, $\mathcal{V}$-isomorphic. ■

4.3 Tensors and cotensors

Proposition 4.3.1 ([Rie14], 3.7.10)**.** *Suppose $\underline{\mathcal{M}}$ and $\underline{\mathcal{N}}$ are tensored and cotensored and we have an adjunction between the underlying categories, $\mathcal{V}$-categories and $F : \mathcal{M} \rightleftarrows \mathcal{N} : G$. Then the data of any of the following determines the other* (*i*) *A $\mathcal{V}$-adjunction* $\underline{\mathcal{N}}(Fm, n) \cong \underline{\mathcal{M}}(m, Gn)$.

(*ii*) *A $\mathcal{V}$-functor F together with natural isomorphisms* $F(v \otimes m) \cong v \otimes Fm$

(*iii*) *a $\mathcal{V}$-functor G together with natural isomorphisms* $G(n^v) \cong G(n)^v$

Proof. We first observe that the desired equivalence of data between (i) and (iii) is immediate from that of (i) and (ii) by passing to the opposite categories.

Given a $\mathcal{V}$-adjunction $\underline{\mathcal{N}}(Fm, n) \cong \underline{\mathcal{M}}(m, Gn)$, the desired functor is a part of the adjunction. The naturality of the adjunction and the tensor products give the natural isomorphisms

$$\underline{\mathcal{N}}(F(v \otimes m), n) \cong \underline{\mathcal{M}}(v \otimes m, Gn) \cong \underline{\mathcal{M}}(v, \underline{\mathcal{M}}(m, Gn)) \cong \underline{\mathcal{M}}(v, \underline{\mathcal{N}}(Fm, n)) \cong \underline{\mathcal{N}}(v \otimes Fm, n),$$

so that $F(v \otimes m) \cong v \otimes Fm$. For the other direction, the first step is thus to find a sequence of natural maps $\mathcal{N}(x, y) \to \mathcal{M}(Gx, Gy)$. The given functor G on underlying categories gives the assignments $x \mapsto Gx$ as well as a natural map $\mathcal{V}(*, \mathcal{N}(x, y)) \to \mathcal{V}(*, \mathcal{N}(Gx, Gy))$.

Then we have the following sequence of natural transformations

$$\begin{aligned}
\underline{\mathcal{V}}(v, \underline{\mathcal{N}}(x, y)) &\cong \underline{\mathcal{V}}(*, \underline{\mathcal{N}}(x, y^v)) \\
&\xrightarrow{G} \underline{\mathcal{V}}(*, \underline{\mathcal{M}}(Gx, G(y^v)) \\
&\cong \underline{\mathcal{V}}(*, \underline{\mathcal{N}}(FGx, y^v)) \\
&\cong \underline{\mathcal{V}}(v, \underline{\mathcal{N}}(FGx, y)) \\
&\cong \underline{\mathcal{V}}(v \otimes FGx, y) \\
&\cong \underline{\mathcal{V}}(F(v \otimes Gx), y) \\
&\cong \underline{\mathcal{V}}(*, \mathcal{N}(F(v \otimes Gx), y)) \\
&\cong \underline{\mathcal{V}}(*, \mathcal{M}(v \otimes Gx, Gy)) \\
&\cong \underline{\mathcal{V}}(v, \mathcal{M}(Gx, Gy)).
\end{aligned}$$

Since $\underline{\mathcal{V}} = V$, this sequence corresponds (naturally in x and y) to a morphism $\underline{\mathcal{N}}(x, y) \to \underline{\mathcal{M}}(Gx, Gy)$ in V by the Yoneda Lemma. So G extends to a functor. ∎

Exercise 4.3.2 ([Rie14], 3.7.18)**.** Let $\mathcal{M}$ be cocomplete. Show that the category $\mathcal{M}^{\mathbb{\Delta}^{\text{op}}}$ of simplicial objects in $\mathcal{M}$ is simplicially enriched and tensored, with $(K \otimes X)_n := K_n \cdot X_n$ defined using the copower. Give a formal argument why $\mathcal{M}^{\mathbb{\Delta}}$ is simplicially enriched and cotensored if $\mathcal{M}$ is complete.

Proof. Initially we struggled to prove this, but we found help in the stack exchange answer of Eric Wofsey [Wof16]. We added some omitted pieces of the argument, and make some commentary while providing his (seemingly canonical/well-known) solution.

We need to assign to each pair of simplicial objects in $\mathcal{M}^{\mathbb{\Delta}^{\text{op}}}$ a simplicial set that is compatible with the tensor product defined above:

$$\mathcal{M}^{\mathbb{\Delta}^{\text{op}}}(X \otimes A, B) \cong \mathbf{sSet}(X, \mathcal{M}^{\mathbb{\Delta}^{\text{op}}}(A, B))$$

Taking X to be representable, this becomes:

$$\mathcal{M}^{\mathbb{\Delta}^{\text{op}}}(\Delta^n \otimes A, B) \cong \mathbf{sSet}(\Delta^n, \mathcal{M}^{\mathbb{\Delta}^{\text{op}}}(A, B)) \cong (\mathcal{M}^{\mathbb{\Delta}^{\text{op}}}(A, B))_n.$$

So, if this tensoring is possible, we need to define $\mathcal{M}^{\mathbb{\Delta}^{\text{op}}}(A, B)$ to be the set of natural transformations between $\Delta^n \otimes A$ and B.

To see that this defines a simplicial set, we note that $\hom$ and tensoring with A are both functorial, so a morphism in Δ^{op} is induced functorially from our definition.

A few things that need to be checked are:

1. For each $A \in \mathcal{M}^{\mathbb{\Delta}^{\text{op}}}$, a morphism $\text{Id}_A : * \to \underline{\mathcal{M}}^{\mathbb{\Delta}^{\text{op}}}(A, A)$.
2. For each A, B, C, a morphism, $\circ : \underline{\mathcal{M}}^{\mathbb{\Delta}^{\text{op}}}(B, C) \times \underline{\mathcal{M}}^{\mathbb{\Delta}^{\text{op}}}(A, B) \to \underline{\mathcal{M}}^{\mathbb{\Delta}^{\text{op}}}(A, C)$

3. The compatibility conditions.

We will not check all of them, since that would involve an incredible amount of typing. However, we offer definitions for 1. and 2. derived in the categorical manner that makes it believable that 3. would hold.

Recall that the monoidal unit in $\mathbf{sSet}$ has one degenerate simplex in each dimension made of repeating the single point however many times. Thus, it has a canonical injection into the simplicial set $\underline{\mathcal{M}}^{\Delta^{\mathrm{op}}}(A, A)$.

For the composition, we write out:

$$(\underline{\mathcal{M}}^{\Delta^{\mathrm{op}}}(B, C))_n \cong \mathcal{M}^{\Delta^{\mathrm{op}}}(\Delta^n \otimes B, C)$$

But we recall that the definition of $\Delta^n \otimes B$ was given levelwise as the copower, which is a coproduct over copies of B. Thus, a natural transformation $\Delta^n \otimes B \Rightarrow C$ particularly induces a natural transformation $B \Rightarrow C$. So, we have can just define the composition map (level-wise) as the composition of the natural transformations (taking the induced natural transformation in the first set) in:

$$\mathcal{M}^{\Delta^{\mathrm{op}}}(\Delta^n \otimes B, C) \times \mathcal{M}^{\Delta^{\mathrm{op}}}(\Delta^n \otimes A, B) \to \mathcal{M}^{\Delta^{\mathrm{op}}}(B, C) \times \mathcal{M}^{\Delta^{\mathrm{op}}}(\Delta^n \otimes A, B) \to \mathcal{M}^{\Delta^{\mathrm{op}}}(\Delta^n \otimes A, C).$$

From these definitions, one can believe that the compatibility conditions will be satisfied.

Note that we still need to verify that the tensoring and enrichment are consistent over all of $\mathbf{sSet}$ not just the representable functors. However, this follows from taking colimits and continuity of $\hom$, so indeed, we see that the tensoring and enrichment are consistent for all objects in $\mathbf{sSet}$.

Finally, to examine some generalizations, we note that we only needed cocompleteness of $\mathcal{M}$ to define the copowers, and that this is the only condition that we need to define an enrichment over $\mathbf{sSet}$. In particular, one could use the representables of any functor category into $\mathbf{Set}$ to generate an enrichment in this way. Further, since all functor categories to $\mathbf{Set}$ satisfy the density theorem, this construction does not depend at all on Δ^{op}, but rather just taking any category of presheaves into $\mathbf{Set}$.

The dual of this result shows that $\mathcal{M}^{\Delta}$ is simplicially enriched and cotensored if $\mathcal{M}$ is complete.

■

Chapter 5

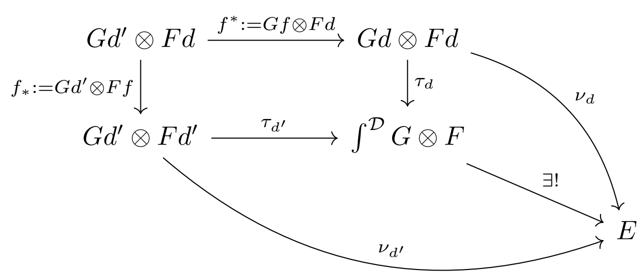
Homotopy Colimits

5.1 First computations

Definition 5.1.1 ([Rie14], 4.1.1)**.** Let $\mathcal{M}$ be a $\mathcal{V}$-tensored category where $-\otimes- : \mathcal{V}\times\mathcal{M}\to\mathcal{M}$ is the tensor. Let $\mathcal{D}$ be any category and let $F:\mathcal{D}\to\mathcal{M}$ and $g:\mathcal{D}^{op}\to\mathcal{V}$ be functors. The **functor tensor product** of F with G is the coend of $G\otimes F := (-\otimes-)\circ(G,F):\mathcal{D}^{\mathrm{op}}\times\mathcal{D}\to\mathcal{M}$.

$$G\otimes_{\mathcal{D}} F := \int^{\mathcal{D}} G\otimes F.$$

Now recall from the definition of coend 1.2.3, this is a pair $(\int^{\mathcal{D}} G\otimes F,\tau)$ where $\int^{\mathcal{D}} G\otimes F$ is an object in $\mathcal{M}$ and τ is a dinatural transformation from H to constant functor at $\int^{\mathcal{D}} G\otimes F$ such that if (E,ν) is any other such pair we get unique morphism from $\int^{\mathcal{D}} G\otimes F$ to E satisfying,



Now one notices that the above coend is nothing but the co-equalizer,

$$G\otimes_{\mathcal{D}} F = \mathrm{coeq}\left(\ \textstyle\coprod_{f:d\to d'} Gd'\otimes Fd \underset{f_*}{\overset{f^*}{\rightrightarrows}} \coprod_d Gd\otimes Fd\ \right)$$

Functor tensor products are useful (and generally computable), and later the goal will be to reduce the following "fattened up" version of the functor tensor product to something skinnier and computable.
Exercise 5.1.2 ([Rie14], 4.1.8, modified slightly)**.** Let $X = X^{-\bullet}\colon \mathbb{\Delta}^{\mathrm{op}}\times\mathbb{\Delta}^{\mathrm{op}}\to \mathbf{Top}$ be a bisimplicial set. Compute $|X|$.

Claim. We claim the following objects are all isomorphic:

- The functor tensor product of X with the Yoneda embedding.
- There are two different ways to think of $|X|$ as a simplicial object in **sTop** by currying, which have their own iterated geometric realizations.
- If $D\colon \mathbb{\Delta}\to\mathbb{\Delta}\times\mathbb{\Delta}$ is the diagonal functor, XD is a simplicial set with its geometric realization $|XD|$.

Computation. First, we compute $\mathrm{Lan}_\Delta(\Delta^\bullet)$, the left Kan extension of the Yoneda embedding along the diagonal functor, $D : \mathbb{\Delta} \to \mathbb{\Delta} \times \mathbb{\Delta}$:

$$\begin{aligned}
\mathrm{Lan}_\Delta(\Delta^\bullet)([m],[n]) &\cong \mathbb{\Delta} \times \mathbb{\Delta}(D-, ([m],[n])) \otimes_{\mathbb{\Delta}} \Delta^\bullet \\
&\cong \int^{[k]\in\mathbb{\Delta}} \mathbb{\Delta} \times \mathbb{\Delta}(([k],[k]),([m],[n])) \cdot \Delta^k \\
&\cong \int^{[k]\in\mathbb{\Delta}} (\mathbb{\Delta}([k],[m]) \times \mathbb{\Delta}([k],[n])) \cdot \Delta^k \\
&\cong \int^{[k]\in\mathbb{\Delta}} (\mathbb{\Delta}([k],[m]) \cdot \Delta^k) \times (\mathbb{\Delta}([k],[n]) \cdot \Delta^k) \\
&\cong \int^{[k]\in\mathbb{\Delta}} (\mathbb{\Delta}([k],[m]) \cdot \Delta^k) \times \int^{[k]\in\mathbb{\Delta}} (\mathbb{\Delta}([k],[n]) \cdot \Delta^k) \\
&\cong \Delta^m \times \Delta^n,
\end{aligned}$$

which is the Yoneda embedding $\Delta \times \Delta$ from $\mathbb{\Delta} \times \mathbb{\Delta} \to \mathbf{Top}^{\mathbb{\Delta}\times\mathbb{\Delta}}$. ∎

Proof. The isomorphism between the objects of 1) and 3) follows from a general lemma concerning weighted colimits. More precisely: The geometric realization of X is the colimit of X weighted by the Yoneda embedding Δ. Then by 6.2.4 and the above computation respectively,

$$|XD| \cong \mathrm{colim}^{\Delta} XD \cong \mathrm{colim}^{\mathrm{Lan}_D \Delta} X \cong \mathrm{colim}^{\Delta\times\Delta} X \cong |X|.$$

The equivalence of 1) and 2) really follows from just the Fubini theorem, but we will be more explicit than is perhaps needed. Currying gives an isomorphism

$$\mathrm{Fun}(\mathbb{\Delta}^{\mathrm{op}} \times \mathbb{\Delta}^{\mathrm{op}}, \mathbf{Top}) \cong \mathrm{Fun}(\mathbb{\Delta}^{\mathrm{op}}, \mathrm{Fun}(\mathbb{\Delta}^{\mathrm{op}}, \mathbf{Top}),$$

and then we can take iterated geometric realization to get a map:

$$\Psi(X) := \mathrm{Fun}(\mathbb{\Delta}^{\mathrm{op}}, \mathrm{Fun}(\mathbb{\Delta}^{\mathrm{op}}, \mathbf{Top}) \xrightarrow{X\mapsto|-|\circ X} \mathrm{Fun}(\mathbb{\Delta}^{\mathrm{op}}, \mathbf{Top}) \xrightarrow{X\mapsto|X|} \mathbf{Top}.$$

And we compute:

$$\begin{aligned}
\Psi(X) \cong \Delta^\bullet \otimes (\Delta^- \otimes X_{\bullet -}) &\cong \int^{c\in\mathbb{\Delta}^{\mathrm{op}}} \mathbb{\Delta}(c,\bullet) \otimes \left(\int^{d\in\mathbb{\Delta}^{\mathrm{op}}} \mathbb{\Delta}(d,-) \otimes X(\bullet,-)\right) \\
&\cong \int^{c\in\mathbb{\Delta}^{\mathrm{op}}} \int^{d\in\mathbb{\Delta}^{\mathrm{op}}} \mathbb{\Delta}(c,\bullet) \otimes \mathbb{\Delta}(d,-) \otimes X(\bullet,-) \\
&\cong \int^{(c,d)\in\mathbb{\Delta}^{\mathrm{op}}\times\mathbb{\Delta}^{\mathrm{op}}} \mathbb{\Delta} \times \mathbb{\Delta}((c,d),(\bullet,-)) \otimes X(\bullet,-) \cong |X|
\end{aligned}$$

using the (co)-continuity of $Z \otimes -$ and the Fubini theorem to perform the required rearrangements, noting that the tensor in **Top** is the cartesian product. ∎

Definition 5.1.3 ([Rie14], 4.2.1)**.** The **two sided simplicial bar construction** produces from

1. A tensored, cotensored, simplicially enriched category $\mathcal{M}$,
2. a small category $\mathcal{D}$,
3. functors $G\colon \mathcal{D}^{\mathrm{op}} \to \mathbf{sSet}$ and $F\colon \mathcal{D} \to \mathcal{M}$,

a simplicial object $B_\bullet(G,\mathcal{D},F)$. The n-simplices of $B_\bullet(G,\mathcal{D},F)$ are defined by the coproduct

$$B_n(G,\mathcal{D},F) = \coprod_{\vec{d}\colon [n]\to\mathcal{D}} Gd_n \otimes Fd_0,$$

where $\vec{d}$ is a shorthand for a sequence $d_0 \to d_1 \to \cdots \to d_n$ of n composable arrows in $\mathcal{D}$. The face and degeneracy maps reindex the coproducts by composing or adding identity arrows as appropriate.

Definition 5.1.4 ([Rie14], 4.2.1)**.** The **bar construction** is the geometric realization of the simplicial bar construction, i.e the functor tensor product of the simplicial bar construction with the Yoneda embedding.
Notation 5.1.5 ([Rie14], 4.2.5)**.** Riehl states that it is convention to write

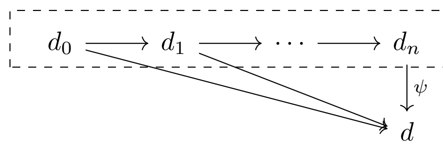
$$B(\mathcal{D}, \mathcal{D}, F) \colon \mathcal{D} \to \mathcal{M} \quad \text{for the functor} \quad d \mapsto B(\mathcal{D}(-, d), \mathcal{D}, F).$$

We will use similar notation for the simplicial bar construction, as in the text.
Exercise 5.1.6 ([Rie14], 4.2.6)**.** Taking $M = \mathbf{Set}$ show that $B_\bullet(\mathcal{D}(-, d), \mathcal{D}, *) \cong N(\mathcal{D}/d)$ so that $B_\bullet(\mathcal{D}, \mathcal{D}, *)$ is naturally isomorphic to the functor $N(\mathcal{D}/-) \colon \mathcal{D} \to \mathbf{sSet}$.

Proof. We want to give a natural bijection $N(\mathcal{D}/d)_n \cong \mathrm{Fun}([n], \mathcal{D}/d) \to B_n(\mathcal{D}(-, d), \mathcal{D}, *)$. We have by definition

$$B_n(\mathcal{D}(-, d), \mathcal{D}, *) = \coprod_{\vec{d} \colon [n] \to \mathcal{D}} \mathcal{D}(d_n, d).$$

We observe the data of a functor $\Psi \colon [n] \to \mathcal{D}/d$ is the following (all triangles commutative),



Defining $\Psi_\square$ as the boxed subsection of the diagram, we have a map $\mathrm{Fun}([n], \mathcal{D}/d) \to B_n(\mathcal{D}(-, d), \mathcal{D}, *)$ where $\Psi \mapsto \psi$ in the $\Psi_\square$ component. This map is a bijection since the data of $\Psi_\square$ together with ψ uniquely determines Ψ, since all the triangles must commute. Naturality is due to the fact that the face and degneracy maps act on Ψ and $\Psi_\square$ in identical ways, namely by adding identities or composing arrows (one of which might be ψ). ■

Definition 5.1.7 ([Rie14, 5.1.3])**.** The homotopy colimit (respectively limit), is the left (right) derived functor of the colimit (limit): $\mathcal{M}^{\mathcal{D}} \to \mathcal{M}$:

$$\mathrm{hocolim}_{\mathcal{D}} := \mathbb{L}\,\mathrm{colim}_{\mathcal{D}} \qquad \mathrm{holim}_{\mathcal{D}} := \mathbb{R} \lim_{\mathcal{D}}$$

Example 5.1.8 ([Rie14] 6.4.5)**.** The double mapping cylinder as a homotopy colimit.
We would like to calculate the homotopy colimit of the following diagram $F : \mathcal{D} \to \mathbf{Top}$:

$$X \xleftarrow{\ f\ } A \xrightarrow{\ g\ } Y$$

This diagram is 1-skeletal since it has no non-trivial compositions, so the corresponding bar construction, $B_\bullet(*, \mathcal{D}, F)$ is also 1-skeletal. So Lemma 4.4.3 of [Rie14] implies that

$$|B_\bullet(*, \mathcal{D}, F)| \cong \Delta^\bullet_{\leq 1} \otimes_{\Delta^{\mathrm{op}}_{\leq 1}} (B_\bullet(*, \mathcal{D}, F))_{\leq 1}.$$

Evaluating the functor tensor product, we get by definition:

$$\Delta^\bullet_{\leq 1} \otimes_{\Delta^{\mathrm{op}}_{\leq 1}} (B_\bullet(*, \mathcal{D}, F))_{\leq 1} = \int^{\Delta^{\mathrm{op}}_{\leq 1}} \Delta^\bullet_{\leq 1} \otimes (B_\bullet(*, \mathcal{D}, F))_{\leq 1}.$$

To evaluate this, we need to calculate the 0 and 1 terms of the simplicial bar construction:

$$B_0(*, \mathcal{D}, F) = \coprod_{\vec{d}:[0] \to \mathcal{D}} *(d_0) \times F(d_0) = X \amalg A \amalg Y$$
$$B_1(*, \mathcal{D}, F) = \coprod_{\vec{d}:[1] \to \mathcal{D}} *(d_1) \times F(d_0) = X^1 \amalg A^f \amalg A^1 \amalg A^g \amalg Y^1.$$

Thus, we want:

$$\int^{\Delta^{op}_{\le 1}} \Delta^\bullet_{\le 1} \otimes (B_\bullet(*, \mathcal{D}, F))_{\le 1} = \text{coeq}\left(\coprod_{f:d\to d'} \Delta^\bullet(d') \times B_\bullet(*, \mathcal{D}, F)(d) \overset{f^*}{\underset{f_*}{\rightrightarrows}} \coprod_{d} \Delta^\bullet(d) \times B_\bullet(*, \mathcal{D}, F)(d) \right)$$

There are three f's indexing the coproduct: two face maps, $\delta_0, \delta_1 : [0] \to [1]$, and one degeneracy map, $\sigma_0 : [1] \to [0]$. We can also write out entirely the object on the right, before we start reducing by the coequalizer:

$$\coprod_d \Delta^\bullet(d) \times B_\bullet(*, \mathcal{D}, F)(d) = (X \amalg A \amalg Y) \amalg \left((X^1 \times I) \amalg (A^f \times I) \amalg (A^1 \times I) \amalg (A^g \times I) \amalg (Y^1 \times I)\right),$$

where the superscripts on the spaces denote which 1-cells in $F(\mathcal{D})$ the spaces arise from. The two maps given by σ_0, s^0, s_0 are (with the maps given for each term in the coproduct):

$$\begin{aligned} s_0 = (\sigma_0)_* :& \Delta^1 \times B_0(*, \mathcal{D}, F) \to \Delta^1 \times B_1(*, \mathcal{D}, F) \\ & (X \times I, A \times I, Y \times I) \mapsto (X^1 \times I, A^1 \times I, Y^1 \times I) \\ s^0 = (\sigma_0)^* :& \Delta^1 \times B_0(*, \mathcal{D}, F) \to \Delta^1 \times B_1(*, \mathcal{D}, F) \\ & (X \times I, A \times I, Y \times I) \mapsto (X, A, Y) \end{aligned}$$

This gives the reductions squishing the $X^1 \times I$, $Y^1 \times I$, and $A^1 \times I$ to just the X, Y, A of $B_0 \times \Delta^0$. Next, we have the two maps given by $\delta_0 : 0 \mapsto 0$, d_0, d^0 :

$$\begin{aligned} d_0 = (\delta_0)^* :& \Delta^0 \times B_1(*, \mathcal{D}, F) \to \Delta^0 \times B_0(*, \mathcal{D}, F) \\ & (X^1, A^f, A^1, A^g, Y^1) \mapsto (X, A, A, A, Y) \\ d^0 = (\delta_0)_* :& \Delta^0 \times B_1(*, \mathcal{D}, F) \to \Delta^1 \times B_1(*, \mathcal{D}, F) \\ & (X^1, A^f, A^1, A^g, Y^1) \mapsto (X^1 \times \{0\}, A^f \times \{0\}, A^1 \times \{0\}, A^g \times \{0\}, Y^1 \times \{0\}) \end{aligned}$$

And finally, we have the maps given by $\delta_1 : 0 \mapsto 1$, d_1, d^1:

$$\begin{aligned} d_1 = (\delta_1)^* :& \Delta^0 \times B_1(*, \mathcal{D}, F) \to \Delta^0 \times B_0(*, \mathcal{D}, F) \\ & (X^1, A^f, A^1, A^g, Y^1) \mapsto (X, X, A, Y, Y) \\ d^1 = (\delta_1)_* :& \Delta^0 \times B_1(*, \mathcal{D}, F) \to \Delta^1 \times B_1(*, \mathcal{D}, F) \\ & (X^1, A^f, A^1, A^g, Y^1) \mapsto (X^1 \times \{1\}, A^f \times \{1\}, A^1 \times \{1\}, A^g \times \{1\}, Y^1 \times \{1\}). \end{aligned}$$

On the $X^1 \times I, Y^1 \times I$ that we have already squished down with the σ_0 identifications, the identifications associated with δ_1 attach them to the ends of the un-squished cylinders: $A^f \times I$ and $A^g \times I$. Then the identifications associated with δ_0 attach $A^1 \times I$, which is also squished down by σ_0, to the bases of the two cylinders. These identifications together give the mapping cylinder.

Remark. A remark or two on the consequences of the above calculation taken from commentary at the beginning of [Dug08]. Firstly, this forms the homotopy pushout of the diagram $\mathcal{D}$ definitionally. It is worth noting that collapsing the two additional cylinders, $A^f \times I$ and $A^g \times I$ down onto A yields the traditional pushout of the two spaces. Every such homotopy colimit will admit a map down to the normal colimit given by collapsing all of the inserted homotopies.

What we have done practically is created a pushout, where the traditional identification, $f(a) = g(a)$, is replaced by a homotopy, H, given by moving across the double mapping cylinder.

Chapter 6

Weighted Limits and Colimits

6.1 The Grothendieck construction

Definition 6.1.1 ([Rie14], nlab)**.** Let $\mathcal{B}$ be a small category. A functor $F : \mathcal{B} \to \mathcal{C}$ is called a **discrete right fibration** if for every object $b \in \mathcal{B}$, for every morphism $f : c \to c'$ in B, and for every $b' \in F^{-1}(c')$ there exists a unique $\tilde{f} : b \to b'$ such that $F(\tilde{f}) = f$

Construction 6.1.2 ([Rie14], 7.1.9, following paragraphs)**.** Given a presheaf $W : \mathcal{C}^{\text{op}} \to \mathbf{Set}$ the **contravariant Grothendieck construction** produces a functor $\mathbf{el}W \to \mathcal{C}$. By definition, objects in the category $\mathbf{el}W$ are pairs $(c \in \mathcal{C}, x \in Wc)$, and morphisms $(c, x) \to (c, x')$ are arrows $f : c \to c'$ in $\mathcal{C}$ such that $Wf : Wc' \to Wc$ takes x' to x. Note that the forgetful functor $\Sigma : \mathbf{el}W \to \mathcal{C}$ is a discrete right fibration.

Given a discrete right fibration $F : \mathcal{B} \to \mathcal{C}$ define a functor $W : \mathcal{C}^{\text{op}} \to \mathbf{Set}$ by taking Wc to be the fiber $F^{-1}(c)$ on objects $c \in \mathcal{C}$, and for morphisms $f : c \to c'$, $W(f) : F^{-1}(c') \to F^{-1}(c)$ is given by $(\mathcal{W}(F)(f))(b') = b$ such that $\tilde{f} : b \to b'$ is the unique lift of f with co-domain b'.

Exercise 6.1.3 ([Rie14], 7.1.9)**.** Verify the above constructions define an equivalence between the category $\mathbf{Set}^{\mathcal{C}^{\text{op}}}$ and the full subcategory of $\mathbf{Cat}/\mathcal{C}$ of discrete right fibrations over $\mathcal{C}$.

Proof. The strategy is

1. Extend the definitions above to be functorial,
2. Define the requisite natural transformations for an equivalence,
3. Perform the requisite diagram chasing and verify some other properties as needed.

1) Let $\mathcal{D}$ be the full subcategory of $\mathbf{Cat}/\mathcal{C}$ of discrete right fibrations over $\mathcal{C}$. We define $\mathfrak{Q} : \mathbf{Set}^{\mathcal{C}^{\text{op}}} \to \mathcal{D}$ on objects (functors) by the construction Σ above. That is, $\mathfrak{Q}F$ is the forgetful functor $\mathbf{el}F \to \mathcal{C}$. From a morphism (natural transformation) $\eta : F \Rightarrow G$ we define a functor $\mathfrak{Q}\eta : \mathfrak{Q}F \to \mathfrak{Q}G$ by the following formulas on objects c and morphisms $f : (c, x) \to (c', x')$ in $\mathbf{el}F$:

$$\mathfrak{Q}\eta(c, x) := (c, \eta_c(x)) \qquad \text{and} \qquad \mathfrak{Q}\eta(f : (c, x) \to (c', x')) := f : (c, \eta_c(x)) \to (c', \eta_{c'}(x')).$$

That f is actually a morphism $(c, \eta_c(x)) \to (c', \eta_{c'}(x'))$ in $\mathbf{el}G$ requires that $Gf(\eta'_c(x')) = \eta_c(x)$. Naturality of η gives $Gf(\eta'_c(x')) = \eta_c(Ff(x'))$ and we have $Ff(x') = x$ since f was a morphism in $\mathbf{el}F$. Functoriality of $\mathfrak{Q}$ is by definition of composition of natural transformations. It remains to check that $\mathfrak{Q}G \circ \mathfrak{Q}\eta = \mathfrak{Q}F$, so that $\mathfrak{Q}\eta$ is a morphism in $\mathcal{D}$. This follows since the functor $\mathfrak{Q}\eta$ is constant on the objects and morphisms as elements of $\mathcal{C}$, and the functors $\mathfrak{Q}G, \mathfrak{Q}F$ are forgetful. For example $\mathfrak{Q}F(f) = f = (\mathfrak{Q}G \circ \mathfrak{Q}\eta)(f)$.

Next, we define a functor $\mathcal{W} : \mathcal{D} \to \mathbf{Set}^{\mathcal{C}^{\text{op}}}$ on objects (functors) by the construction W above. From a morphism (functor) $K : F \to G$ in $\mathcal{D}$ (so $F = GK$) we define a morphism (natural transformation) $\mathcal{W}K : \mathcal{W}F \Rightarrow \mathcal{W}G$ with legs $(\mathcal{W}K)_c : F^{-1}(c) \to G^{-1}(c)$ by $(\mathcal{W}K)_c(e) := Ke$. This definition is valid since $F = GK$ implies $Ke \in G^{-1}(c)$ and is functorial in K.

2, 3) We first define the natural isomorphism $\Phi\colon \mathbb{Q}\mathscr{W} \Rightarrow \mathbb{1}_{\mathcal{D}}$, which consists of a functor $\Phi_F\colon \mathbf{el}\mathscr{W}F \to \mathcal{E}$ for each $F\colon \mathcal{E} \to \mathcal{C} \in \mathcal{D}$, so that the following diagrams commute for all $K\colon F \to F' \in \mathcal{D}$:

$$
\begin{array}{ccc}
 & \mathcal{C} & \\
{\scriptstyle \mathbb{Q}\mathscr{W}F}\nearrow & & \nwarrow{\scriptstyle F} \\
\mathbf{el}\mathscr{W}F & \xrightarrow{\Phi_F} & \mathcal{E}
\end{array}
\qquad \text{and} \qquad
\begin{array}{ccc}
\mathbb{Q}\mathscr{W}F & \xrightarrow{\mathbb{Q}\mathscr{W}K} & \mathbb{Q}\mathscr{W}F' \\
\downarrow{\scriptstyle \Phi_F} & & \downarrow{\scriptstyle \Phi_{F'}} \\
F & \xrightarrow{K} & F'
\end{array}
$$

(in $\mathbf{Cat}$ and $\mathcal{D}$ respectively) along with the further requirement that Φ_F is an *isomorphism* for each F.

We first unpack the definition of the category $\mathbf{el}\mathscr{W}F$. The objects are pairs (c, e) where $c \in \mathcal{C}$ and $e \in \mathscr{W}F(c) = F^{-1}(c)$. Thus e is an object of $\mathcal{E}$. And morphisms are $g : (c, e) \to (c', e')$ where $g : c \to c'$ is a morphism in $\mathcal{C}$ such that $\mathscr{W}F(g) : \mathscr{W}F(c') \to \mathscr{W}F(c)$ which takes e' to e. For $(c, e) \in \mathbf{el}\mathscr{W}F$ and $g : (c, e) \to (c', e') \in \mathbf{el}\mathscr{W}F$ we define $\Phi_F(c, x) := x$. The morphism $\Phi_F(g)\colon e \to e'$ is defined to be $\tilde{g}$ where $\tilde{g}$ is the unique lift of g along F with co-domain e', $\tilde{g}$ is indeed a morphism from e to e' is guaranteed by the Grothendieck construction. Functoriality of this definition follows from the uniqueness of $\tilde{g}$, specifically $\Phi_F(gf) = \tilde{gf}$, but $F(\tilde{g}\tilde{f}) = F(\tilde{g})F(\tilde{f}) = gf$ and from the uniqueness $\tilde{gf} = \tilde{g}\tilde{f}$. Now commutativity of the left-hand diagram above is tautological (analogous to that of $\mathbb{Q}\eta$ in part 1).

Now we check the commutivity of the right hand diagram. Since we checked the commutativity of the left hand diagram, it is enough to check the commutativity of following diagram,

$$
\begin{array}{ccc}
\mathbf{el}\mathscr{W}F & \xrightarrow{\mathbb{Q}(\mathscr{W}K)} & \mathbf{el}\mathscr{W}F' \\
\downarrow{\scriptstyle \Phi_F} & & \downarrow{\scriptstyle \Phi_{F'}} \\
\mathcal{E} & \xrightarrow{K} & \mathcal{E}'.
\end{array}
$$

Let $(c, e) \in \mathbf{el}\mathscr{W}F$, then $\mathbb{Q}(\mathscr{W}K)(c, e) = (c, (\mathscr{W}K)_c(e))$. Recall that $(\mathscr{W}K)$ is a natural transformation whose legs are given by $(\mathscr{W}K)_c(e) = K(e)$. Thus the top-right maps (c, e) to $\Phi_{F'}(c, K(e)) = K(e)$. Now left-bottom maps (c, e) to $K(e)$. Thus the functors agree on objects. Now let $g : (c, e) \to (c', e')$ be a morphism in $\mathbf{el}\mathscr{W}F$, then

$$\mathbb{Q}(\mathscr{W}K)(g : (c, e) \to (c', e')) = g : (c, (\mathscr{W}K)_c(e)) \to (c', (\mathscr{W}K)'_c(e'))$$

But since $(\mathscr{W}K)_c(e) = K(e)$ we get $g : (c, Ke) \to (c', Ke')$ Now taking $\Phi_{F'}$ we get g maps to $\tilde{g}_{Ke'}$, the unique lift of g along F' with co-domain Ke'. Whereas left-bottom maps to $K(\tilde{g}_e)$. Observe that $F'(K(\tilde{g}_e)) = F(\tilde{g}_e) = g$. Thus $K(\tilde{g}_e)$ is a lift of g along F' with co-domain $K(e')$. Thus from uniqueness we get $K(\tilde{g}_e) = \tilde{g}_{Ke'}$. Thus the functors agree on morphisms hence the diagram commutes.

Now we will show that Φ_F is an isomorphism by constructing its inverse, $\Phi_F^{-1} : \mathcal{E} \to \mathbf{el}\mathscr{W}F$, by defining $\Phi_F^{-1}(e) := (Fe, e)$. The morphism $\Phi_F^{-1}(g)\colon (Fe, e) \to (Fe', e')$ is defined to be Fg, valid as a morphism in $\mathbf{el}\mathscr{W}F$ since Fg is the *unique* arrow lifting $Fg\colon Fe \to Fe'$ with codomain e' (by the discrete fibration property) and therefore $\mathscr{W}Fg(e) = e'$ by definition of $\mathscr{W}$. With these definitions, functoriality of Φ_F^{-1} is inherited from that of F.

Now $\Phi_F^{-1}\Phi_F(c, e) = \Phi_F^{-1}(e) = (Fe, e) = (c, e)$, and $\Phi_F^{-1}\Phi_F(g) = \Phi_F^{-1}(\tilde{g}) = F(\tilde{g}) = g$. One checks the other direction in similar fashion. Thus $\Phi : \mathbb{Q}\mathscr{W} \Rightarrow \mathbb{1}_{\mathcal{D}}$ is a natural isomorphism.

The natural isomorphism $\Psi\colon \mathbb{1}_{\mathbf{Set}^{\mathcal{C}^{\mathrm{op}}}} \Rightarrow \mathscr{W}\mathbb{Q}$ consists of a natural isomorphism $\Psi_F\colon F \Rightarrow \mathscr{W}\mathbb{Q}F$ for each $F \in \mathbf{Set}^{\mathcal{C}^{\mathrm{op}}}$. In turn, $(\Psi_F)_c := \Psi_{Fc}$ consists of a bijection $Fc \to \mathscr{W}\mathbb{Q}Fc$. Finally, the following diagrams must commute for any natural transformation $\eta\colon F \Rightarrow G$ and any morphism $f\colon c \to c'$ in $\mathcal{C}$:

$$
\begin{array}{ccc}
F & \xRightarrow{\eta} & G \\
{\scriptstyle \Psi_F}\Downarrow & & \Downarrow{\scriptstyle \Psi_G} \\
\mathscr{W}\mathbb{Q}F & \xRightarrow{\mathscr{W}\mathbb{Q}\eta} & \mathscr{W}\mathbb{Q}G
\end{array}
\qquad
\begin{array}{ccc}
Fc' & \xrightarrow{Ff} & Fc \\
{\scriptstyle \Psi_{Fc'}}\downarrow & & \downarrow{\scriptstyle \Psi_{Fc}} \\
\mathscr{W}\mathbb{Q}Fc' & \xrightarrow{\mathscr{W}\mathbb{Q}Ff} & \mathscr{W}\mathbb{Q}Fc
\end{array}
$$

Unpacking the object $\mathscr{W}\mathbb{Q}Fc = (\mathbb{Q}F)^{-1}(c) = \{(c, x) | x \in Fc\} \subset \mathbf{el}F$ we define $\Psi_{Fc}(x) = (c, x)$, which is immediately a bijection. We check the commutativity of the right hand diagram first. The top compositite maps $x' \in Fc'$ to $(c, Ff(x'))$ and the bottom composite is $\mathscr{W}\mathbb{Q}Ff(c', x')$

Worth noting is that the order of composition matters a lot here. Specifically, the bottom composite is $((\boldsymbol{\mathscr{W}}(\mathfrak{Q}(F)))(f))(c', x')$. Evaluating this, we get $\boldsymbol{\mathscr{W}}(\mathfrak{Q}F)(f)(c', x')$ is the domain of unique lift of f along $\mathfrak{Q}F$ with co-domain (c', x'). Now observe that $f : (c, x) \to (c', x')$ is a morphism in $\mathbf{el}F$ such that $Ff(x') = x$. Thus f is the required unique lift and hence $\boldsymbol{\mathscr{W}}(\mathfrak{Q}F)(f)(c', x') = (c, x) = (c, Ff(x'))$. Thus the right diagram commutes.

Now since we checked the right hand diagram, it is enough to check following diagram,

$$\begin{array}{ccc} Fc & \xrightarrow{\eta_c} & Gc \\ {\scriptstyle \Psi_{Fc}}\downarrow & & \downarrow{\scriptstyle \Psi_{Gc}} \\ \boldsymbol{\mathscr{W}}(\mathfrak{Q}F)(c) & \xrightarrow{(\boldsymbol{\mathscr{W}}\mathfrak{Q}\eta)_c} & \boldsymbol{\mathscr{W}}(\mathfrak{Q}G)(c) \end{array} .$$

On elements, we claim the above diagram expands to:

$$\begin{array}{ccc} x & \xmapsto{\eta_c} & \eta_c(x) \\ {\scriptstyle \Psi_{Fc}}\downarrow & & \downarrow{\scriptstyle \Psi_{Gc}} \\ (c, x) & \xmapsto{(\boldsymbol{\mathscr{W}}\mathfrak{Q}\eta)_c} & (c, \eta_c(x)) \end{array} .$$

The composite of the top left and right arrows is from their respective definitions, whereas the bottom composite follows has the correct action from the equation $(c, x) \mapsto (\mathfrak{Q}\eta)(c, x) = (c, \eta_c(x))$. Thus $\Psi : \mathbb{1}_{\mathcal{D}} \Rightarrow \boldsymbol{\mathscr{W}}\mathfrak{Q}$ is a natural isomorphism. ∎

6.2 Weighted Limits and Colimits

Exercise 6.2.1 ([Rie14], 7.2.2)**.** Express a cokernel pair, i.e. a pushout of $f : a \to b$ along itself as a weighted colimit.

Proof. Let $f : \mathbb{2} \to \mathcal{M}$[1] be the functor with image $f : a \to b$ and let $W : \mathbb{2}^{\mathrm{op}} \to \mathbf{Set}$ have image $* \amalg * \to *$. Then $\mathrm{colim}^W f$ satisfies,

$$\mathcal{M}(\mathrm{colim}^W f, m) \cong \mathbf{Set}^{\mathbb{2}^{\mathrm{op}}}(W, \mathcal{M}(f, m)).$$

A natural transformation, $W \Rightarrow \mathcal{M}(f, m)$ has two legs. The leg of the natural transformation corresponding to 1 is the set map, $* \to \mathcal{M}(a, m)$ (i.e. isomorphic to a morphism, $a \to m$) and the leg corresponding to 2 is the set map $* \amalg * \to \mathcal{M}(b, m)$ (i.e. isomorphic to two morphisms, $b \to m$). Furthermore, the naturality statement insists that precomposing either of these maps by f yields the given map $a \to m$. Thus, morphisms $\mathrm{colim}^W f \to m$ are in bijection with pairs of morphisms that have common precomposite with f. This is precisely the definition of the pushout:

$$\begin{array}{ccccc} a & \xrightarrow{f} & b & & \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle p} & \searrow & \\ b & \xrightarrow{q} & \mathrm{colim}^W(f) & & \downarrow \\ & \searrow & & \overset{\exists!}{\dashrightarrow} & m \end{array}$$

Then the morphisms p and q satisfying this pushout are the definition of a cokernel pair. ∎

Exercise 6.2.2 ([Rie14], 7.6.5)**.** Prove, using the defining universal property, that in a co-tensored $\mathcal{V}$-category, colimits with arbitrary weights preserve (pointwise) tensors. Note this implies a complementary result to lemma [Rie14, 3.8.3]: in a tensored and cotensored simplicial category $\mathcal{M}$, geometric realization preserve both pointwise tensors and the tensors defined in [Rie14, 3.8.2].

[1] For convenience, we will let 1 denote the initial object and 2 denote the terminal object in this category.

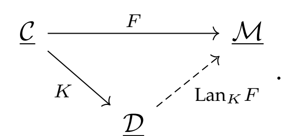
Proof. Let $F : \underline{\mathcal{D}} \to \underline{\mathcal{M}}$ and $W : \underline{\mathcal{D}}^{\mathrm{op}} \to \underline{\mathcal{M}}$ be $\mathcal{V}-$functors. For $v \in \mathcal{V}$, we want to show that $v \otimes \operatorname{colim}^W F \cong \operatorname{colim}^W(v \otimes F-)$. Consider the following chain of natural isomorphisms:

$$\begin{aligned}\underline{\mathcal{M}}(v \otimes \operatorname{colim}^W F, m) &\cong \underline{\mathcal{M}}(\operatorname{colim}^W F, m^v)\\ &\cong \underline{\mathcal{V}}^{\mathcal{D}^{\mathrm{op}}}(W, \underline{\mathcal{M}}(F-, m^v))\\ &\cong \underline{\mathcal{V}}^{\mathcal{D}^{\mathrm{op}}}(W, \underline{\mathcal{M}}(v \otimes F-, m))\\ &\cong \underline{\mathcal{M}}(\operatorname{colim}^W(v \otimes F-), m).\end{aligned}$$

Here the first and third isomorphisms are tensor-cotensor adjunction and the second and fourth isomorphisms are from the universal property of colimits. Thus we conclude $v \otimes \operatorname{colim}^W F \cong \operatorname{colim}^W(v \otimes F-)$ as required. ■

Exercise 6.2.3 ([Rie14] 7.6.8)**.** Suppose $\mathcal{M}$ is $\mathcal{V}$-bicomplete. Describe the universal property of left and right Kan extensions. Show conversely that a $\mathcal{V}$-functor satisfying the appropriate universal property is a pointwise Kan extension in the sense previously defined.

Proof. All categories, functors, and natural transformations in this problem are appropriately enriched over $\mathcal{V}$, and $\underline{\mathcal{C}}$ is assumed to be small. For reference, the diagram of categories and functors is



We claim that a Kan extension of F along K, defined in Riehl as

$$\mathrm{Lan}_K F(d) := \operatorname{colim}^{\mathcal{D}(K-,d)} F \cong \mathcal{D}(K, d) \otimes_{\underline{\mathcal{C}}} F$$

is equivalently a representing object for the functor

$$\underline{\mathcal{M}}^{\underline{\mathcal{C}}}(F, - \circ K) \colon \underline{\mathcal{M}}^{\underline{\mathcal{D}}} \to \underline{\mathcal{V}}.$$

Existence is due to $\mathcal{V}$-bicompleteness of $\mathcal{M}$, so it remains to verify the definition of $\mathrm{Lan}_K F(d)$ satisfies the purported universal property. We do so by explaining the following sequence of isomorphisms natural in $G \in \underline{\mathcal{M}}^{\underline{\mathcal{D}}}$.

$$\begin{aligned}\underline{\mathcal{M}}^{\underline{\mathcal{D}}}(\mathrm{Lan}_K F, G) &:= \int_{d \in \underline{\mathcal{D}}} \underline{\mathcal{M}}(\mathrm{Lan}_K F(d), Gd) \cong \int_{d\in\underline{\mathcal{D}}} \underline{\mathcal{M}}\left(\int^{c \in \underline{\mathcal{C}}} \underline{\mathcal{D}}(Kc, d) \otimes Fc, Gd\right)\\ &\cong \int_{d\in\underline{\mathcal{D}}}\int_{c \in \underline{\mathcal{C}}} \underline{\mathcal{M}}(\underline{\mathcal{D}}(Kc, d) \otimes Fc, Gd) \cong \int_{d\in\underline{\mathcal{D}}}\int_{c\in\underline{\mathcal{C}}} \underline{\mathcal{M}}\left(Fc, Gd^{\underline{\mathcal{D}}(Kc,d)}\right)\\ &\cong \int_{c\in\underline{\mathcal{C}}} \underline{\mathcal{M}}\left(Fc, \int_{d\in\underline{\mathcal{D}}} Gd^{\underline{\mathcal{D}}(Kc,d)}\right) \cong \int_{c\in\underline{\mathcal{C}}} \underline{\mathcal{M}}(Fc, GKc) =: \underline{\mathcal{M}}^{\underline{\mathcal{C}}}(F, GK).\end{aligned}$$

The isomorphisms on the first line are definitional. The second line uses the cocontinuity of enriched hom in the first variable and then the enriched hom-tensor-cotensor adjunction along with the Fubini theorem. We finish by using the continuity of hom in the second variable and the Yoneda Lemma. The converse is the Yoneda Lemma. For completeness: Suppose L is a functor which also satisfies the universal property of the Kan extensions given above. The Enriched Yoneda Lemma (technically the pre-requisite lemma 3.5.12) in Riehl applied to $\underline{\mathcal{M}}^{\underline{\mathcal{D}}}$ along with the natural isomorphisms

$$\underline{\mathcal{M}}^{\underline{\mathcal{D}}}(\mathrm{Lan}_K F, -) \cong \underline{\mathcal{M}}^{\underline{\mathcal{C}}}(F, - \circ K) \cong \underline{\mathcal{M}}^{\underline{\mathcal{D}}}(L, -)$$

imply $\mathrm{Lan}_K F \cong L$. Dually, we assert a right Kan extension of F along K is a representation for the functor $\underline{\mathcal{M}}^{\underline{\mathcal{C}}}(- \circ K, F) \colon \underline{\mathcal{M}}^{\underline{\mathcal{C}}} \to \underline{\mathcal{V}}$, and omit the proof which is similar to the above. We observe that it is possible to have a Kan extension (representing object) which is not defined in the "pointwise sense" only if the target $\mathcal{M}$ is not $\mathcal{V}$-bicomplete. ■

Exercise 6.2.4 ([Rie14], 8.1.5)**.** If $\mathcal{M}$ is a tensored and cotensored $\mathcal{V}$-category and $K : \underline{\mathcal{C}} \to \underline{\mathcal{D}},\ F : \underline{\mathcal{D}} \to \underline{\mathcal{M}}$, and $W : \underline{\mathcal{D}}^{\mathrm{op}} \to \underline{\mathcal{V}}$ are $\mathcal{V}$-functors, then:

$$\mathrm{colim}^{\mathrm{Lan}_K W}(F) \cong \mathrm{colim}^{W}(FK)$$

Proof. We directly calculate, using co-continuity,the coYoneda lemma, and Fubini as appropriate:

$$\begin{aligned}
\mathrm{colim}^{\mathrm{Lan}_K W}(F) &\cong \mathrm{Lan}_K W \otimes_{\underline{\mathcal{D}}} F \\
&\cong \int^{d \in \underline{\mathcal{D}}} \mathrm{Lan}_K W(d) \otimes F(d) \\
&\cong \int^{d \in \underline{\mathcal{D}}} (\underline{\mathcal{D}}(K-, d) \otimes_{\underline{\mathcal{D}}} W) \otimes F(d) \\
&\cong \int^{d \in \underline{\mathcal{D}}} \int^{c \in \underline{\mathcal{C}}} \underline{\mathcal{D}}(Kc, d) \otimes Wc \otimes Fd \\
&\cong \int^{c \in \underline{\mathcal{C}}} Wc \otimes \int^{d \in \underline{\mathcal{D}}} \underline{\mathcal{D}}(Kc, d) \otimes Fd \\
&\cong \int^{c \in \underline{\mathcal{C}}} Wc \otimes FKc \\
&\cong W \otimes_{\underline{\mathcal{C}}} FK \\
&\cong \mathrm{colim}^{W} FK.
\end{aligned}$$

This proof is very similar to 5.1.2 part II (the part which does not use this result). ■

Corollary 6.2.5 (8.1.6 [Rie14])**.** *Let $\mathcal{M}$ be a tensored and cotensored simplicial category. Then the geometric realization, $|-| : \mathcal{M}^{\Delta^{op}} \to \mathcal{M}$ preserves the simplicial tensor defined pointwise by $(K \otimes X)_n := K_n \cdot X_n$.*

Proof. Riehl gives the string of isomorphisms without explanation:

$$K \otimes |X| \cong (K_\bullet \otimes_{\mathbb{\Delta}} \Delta^\bullet) \otimes (\Delta^\bullet \otimes_{\mathbb{\Delta}^{\mathrm{op}}} X_\bullet) \cong (\Delta^\bullet \times \Delta^\bullet) \otimes_{\mathbb{\Delta}^{\mathrm{op}} \times \mathbb{\Delta}^{\mathrm{op}}} (\sqcup_{K_\bullet} X_\bullet) \cong \Delta^\bullet \otimes_{\mathbb{\Delta}^{\mathrm{op}}} (\sqcup_{K_\bullet} X_\bullet) \cong |K \otimes X|$$

We observe the first is definitional, the second is an application of Fubini, the third is an application of 5.1.2 (using the above result) and the definition of the cotensor, and the last is again definitional (4.3.2). ■

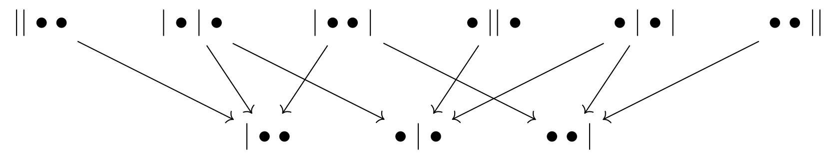
Exercise 6.2.6 ([Rie14] 8.5.11)**.** Write $\mathbb{2}_2$ for the category $\bullet \rightrightarrows \bullet$. Show the functor $F \colon \mathbb{2}_2 \to \mathbb{\Delta}$ with image $[1] \rightrightarrows [0]$ is not homotopy final.

Proof. The strategy is to compute $|N([2]/F)| \simeq S^1$, so that F is not homotopy final. The first step is to analyze the structure of the category $[2]/F$. Objects of $[2]/F$ are morphisms $[2] \to [0]$ and $[2] \to [1]$ in $\mathbb{\Delta}^{\mathrm{op}}$. These are the opposites of morphisms $[0], [1] \to [2] \in \mathbb{\Delta}$, i.e order preserving maps from $[0], [1] \to [2]$. We will represent both kinds of objects as 2 dots, implicitly ordered by their position on the page. These dots are the elements $1, 2 \in [2]$ (so we don't write the element 0). If $f \colon [0] \to [2]$ is a map we write a bar after the image of 0. Likewise, for $f \colon [1] \to [2]$ we write bars after the images of 0 and 1. For instance the map $\{0, 1\} = [1] \to [2] = \{0, 1, 2\}$ sending 0 to 0 and $1 \to 2$ has notation $| \bullet \bullet |$.

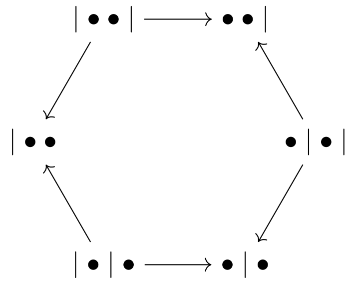
Since the only endomorphisms of $[0], [1]$ in the image of F are identity, any object in $[2]/F$ also has only identity endomorphisms. For morphisms, a morphism $a \to b$ is a map $f \colon [2] \to [2]$ such that either

$$\begin{array}{ccc} [2] & \xrightarrow{f} & [2] \\ \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle b} \\ [0] & \xrightarrow{0 \mapsto 0} & [1] \end{array} \quad \text{or} \quad \begin{array}{ccc} [2] & \xrightarrow{f} & [2] \\ \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle b} \\ [0] & \xrightarrow{0 \mapsto 1} & [1] \end{array} \quad \text{commutes.}$$

Since the only choice is $f = \mathrm{Id}_{[2]}$, these choices correspond exactly to erasing one of the bars in the graphical representation. Since you can only erase one bar, $[2]/F$ has no nontrivial compositions (any composable tuple of n morphisms has at least $n-1$ identity morphisms) and therefore has 1-skeletal nerve and $|N[2]/F| \simeq |N[2]/F|_{\leq 1}$. We have the following diagram of $|N[2]/F|_{\leq 1}$:



which, after contracting the objects || • •, •||• and • • || along their attaching 1-cells, rearranges to

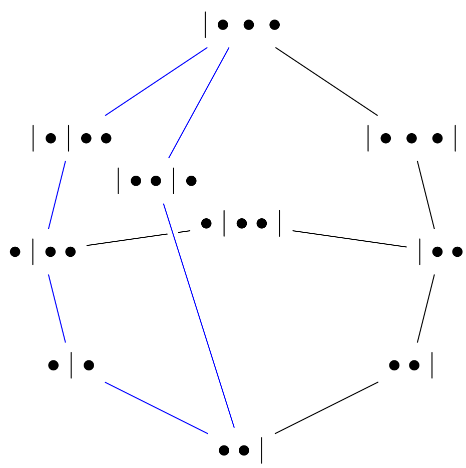


as desired. ■

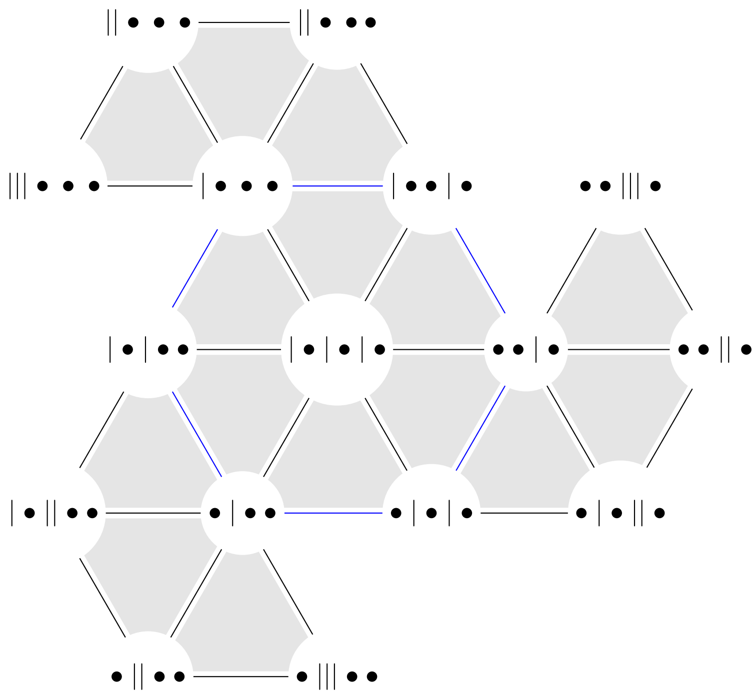
The authors pushed an identical argument through with significantly increased computational effort (The resulting cell structure has 50 0-cells, 108 1-cells, and 60 2-cells) to show the $n = 2$ case of the following conjecture:

Conjecture 6.2.7. *Let* $G_n, n \geq 1$ *be the inclusion of the full subcategory with objects* $[0], [1], \cdots, [n]$ *into* Δ^{op}. *Then* $|N([n+1]/G_n)| \simeq S^n$ *and so none of these functors are homotopy final.*

Idea of proof. As before, we get a "graphical" language for the category $|N([d]/G_n)|$ as pictures of d dots with $1 \leq k < n+1$ separating bars. Morphisms will be erasing some number of bars, leaving at least one. This category is n-skeletal since any $n+1$ tuple of composable morphisms will contain an identity (can't erase more than n bars). At this point we choose $d = [n+1]$, and the difficulty enters because there will be lots of degenerate simplices which add contractible "junk" onto the (complicated) cell structure on the n-sphere. We now show the resolution in $n = 2$ case. With the direction of arrows omitted, part of the 1-skeleton of the 2-sphere we arrive at is:



We would want that the cells in the complete 2-skeleton that attach onto the highlighted hexagon just fill it, but the actual strucure that attaches here is:



(with 2-cells in gray). We see that extra cells appear, but there's still a deformation retraction onto the sphere.

■

Chapter 7

Model Categories

7.1 Weak factorization systems in model categories

Exercise 7.1.1 ([Rie14], 11.1.9)**.** Prove that $\hat{\otimes}$, $\hat{\{\}}$, and $\underline{\hat{\hom}}$ define a two-variable adjunction between the arrow categories by writing down the explicit hom-set bijections

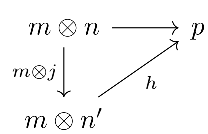
$$\mathcal{P}^{\mathbf{2}}(i\hat{\otimes}j, f) \cong \mathcal{N}^{\mathbf{2}}(j, \widehat{\{i, f\}}) \cong \mathcal{M}^{\mathbf{2}}(i, \underline{\hat{\hom}}(j, f)).$$

A warm up exercise might be in order: Consider $m \otimes n' \xrightarrow{h} p$ and let $n \xrightarrow{j} n'$. Show that the transpose of $m \otimes n \xrightarrow{m\otimes j} m \otimes n' \xrightarrow{h} p$ is the composite $m \xrightarrow{\overline{h}} \underline{\hat{\hom}}(n', p) \xrightarrow{j^*} \underline{\hat{\hom}}(n, p)$, where $\overline{h}$ is the transpose of h.

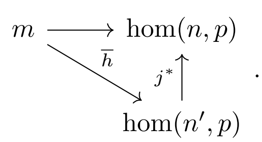
Proof. We first verify the smaller claim. The basic tensor-hom adjunction we are given is that:

$$\mathcal{P}(m \otimes n, p) \cong \mathcal{M}(m, \hom(j, f)).$$

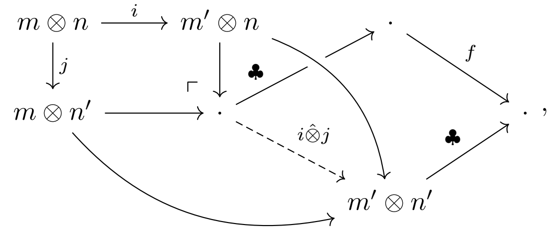
By the naturality of the adjunction we have that the commutation of the triangle



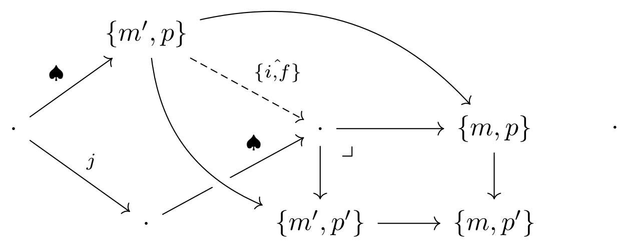
is equivalent to that of the triangle



This gives the conclusion desired by the "warm-up" exercise. We will describe the hom-set bijection between $\hat{\otimes}$ and $\hat{\{\}}$, since the other bijection is similar. A morphism $i\hat{\otimes}j \to f$ is given by a pair of arrows (labeled with ♣) so that the following commutes:



while a morphism $j \to \{i, \hat{f}\}$ is is given by the pair of arrows labeled with ♠ in the commutative diagram:



These diagrams are adjoints of each other, so each commutes if and only if the other does. We conclude morphisms $j \to \{i, \hat{f}\}$ and morphisms $i \hat{\otimes} j \to f$ are in bijection. ■

7.2 Small Object Argument

Definition 7.2.1 ([DS95], 3.1)**.** Given a commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{f} & X \\ \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle p} \\ B & \xrightarrow{g} & Y \end{array} \tag{7.2.1}$$

a **lift** or **lifting** in the diagram is a map $h : B \to X$ such that the resulting diagram with five arrows commute, i.e $g = ph$ and $f = hi$.

Definition 7.2.2 ([DS95], 3.12)**.** A map $i : A \to B$ is said to have the **left lifting property** (LLP) with respect to another map $p : X \to Y$ and p is said to have **right lifting property** (RLP) with respect to i if a lift exists in any diagram of the form 7.2.1.

Given a map $p : X \to Y$ and a set of maps $\mathcal{F} = \{f_i : A_i \to B_i\}_{i \in I}$, we desire to factor p as a composite $X \to X' \to Y$ such that the map $X' \to Y$ has the RLP with respect to all of the maps in $\mathcal{F}$. We can however take $X' = Y$ but we want to find X' as "close" to X as possible. The construction depends on an argument called the "small object argument" that is due to Quillen.

Let $\mathbf{Z}^+ = \{0 \to 1 \to 2 \to \cdots\}$ be the category with objects non-negative integers and a single morphism $i \to j$ for $i \leq j$. Let $\mathcal{C}$ be a category with all small colimits. Given a functor $F : \mathbf{Z}^+ \to \mathcal{C}$ and an object A of $\mathcal{C}$, the natural maps $F(n) \to \operatorname{colim} F$ induce maps $\mathcal{C}(A, F(n)) \to \mathcal{C}(A, \operatorname{colim} F)$. Thus we get a map

$$\operatorname{colim}_n \mathcal{C}(A, F(n)) \to \mathcal{C}(A, \operatorname{colim}_n F(n)). \tag{7.2.2}$$

Definition 7.2.3 ([DS95], 7.14)**.** An object A of $\mathcal{C}$ is said to be **sequentially small** if for every functor $F : \mathbf{Z}^+ \to \mathcal{C}$ the canonical map 7.2.2 is a bijection.

Remark 7.2.4 ([DS95], 7.15). A set is sequentially small if and only if it is finite. An R-module is sequentially small if it has a finite presentation. I.e., it is isomorphic to the cokernel of a map between two finitely generated free R-module. A chain complex $M_\bullet$ is sequentially small if only a finite number of the modules M_k are non-zero, and each M_k has finite presentation.

Proposition 7.2.5. *Suppose an object A of $\mathcal{C}$ is sequentially small. Let $F : \mathbf{Z}^+ \to \mathcal{C}$ be a functor, and let $g : A \to$ colim$_n$ $F(n)$ a morphism in $\mathcal{C}$. Then there is $k \geq 0$ and $g' : A \to F(k)$ such that $g = i_k g'$ where $i_k : F(k) \to$ colim$_n$ $F(n)$ is a natural map.*

Proof. Since A is sequentially small there is $g_1 \in \operatorname{colim}_n \mathcal{C}(A, F(n))$ such that $g_1 \mapsto g$. Since this colimit is in **Set**, there exists $g' \in \mathcal{C}(A, F(k))$ for some $k \geq 0$ such that $g' \mapsto g_1$ under the canonical map $\mathcal{C}(A, F(k)) \to \operatorname{colim}_n(A, F(n))$. By definition, the composition

$$\mathcal{C}(A, F(k)) \to \operatorname{colim}_n \mathcal{C}(A, F(n)) \to \mathcal{C}(A, \operatorname{colim}_n F(n))$$

is given by composing with the natural map $i_k : F(k) \to \operatorname{colim}_n F(n)$. Thus $g = i_k g$. ■

Construction 7.2.6 ([DS95], discussion following 7.15)**.** Let $\mathcal{F} = \{f_i : A_i \to B_i\}_{i \in I}$ be a set of maps and $p : X \to Y$ be a map in a cocomplete category $\mathcal{C}$. For each $i \in I$ consider the set $S(i)$ which contains all pairs of maps (g, h) such that the following square commutes,

$$\begin{array}{ccc} A_i & \xrightarrow{g} & X \\ \downarrow{\scriptstyle f_i} & & \downarrow{\scriptstyle p} \\ B_i & \xrightarrow{h} & Y \end{array} \tag{7.2.3}$$

We define the **Gluing construction** $G^1(\mathcal{F}, p)$ to be the object of $\mathcal{C}$ given by the pushout diagram

$$\begin{array}{ccc} \coprod_{i \in I} \coprod_{(g,h) \in S(i)} A_i & \xrightarrow{+_i +_{(g,h)} g} & X \\ {\scriptstyle \coprod_i \coprod_{(g,h)} f_i}\downarrow & & \downarrow{\scriptstyle i_1} \\ \coprod_{i \in I} \coprod_{(g,h) \in S(i)} B_i & \xrightarrow{j_1} & G^1(\mathcal{F}, p) \end{array}$$

where $+_i +_{(g,h)} g$ is the canonical map induced from all such g. One can think of this construction as gluing a copy of B_i to X along A_i for every commutative diagram of the form 7.2.3. From the universal property of pushout we get a map $p_1 : G^1(\mathcal{F}, p) \to Y$ such that $p_1 i_1 = p$. Now define $G^k(\mathcal{F}, p)$ and p_k for $k > 1$ iteratively by setting $G^k(\mathcal{F}, p) = G^1(\mathcal{F}, p_{k-1})$ and $p_k = (p_{k-1})_1$. We get the commutative diagram,

$$\begin{array}{ccccccccccc} X & \xrightarrow{i_1} & G^1(\mathcal{F}, p) & \xrightarrow{i_2} & G^2(\mathcal{F}, p) & \xrightarrow{i_3} & \cdots & \xrightarrow{i_k} & G^k(\mathcal{F}, p) & \xrightarrow{i_{k+1}} & \cdots \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p_1} & & \downarrow{\scriptstyle p_2} & & & & \downarrow{\scriptstyle p_k} & & \\ Y & \xrightarrow{=} & Y & \xrightarrow{=} & Y & \xrightarrow{=} & \cdots & \xrightarrow{=} & Y & \xrightarrow{=} & \cdots \end{array}$$

Let $G^\infty(\mathcal{F}, p)$, the Infinite Gluing Construction, be the colimit of the upper row in the diagram. Then there are natural maps $i_\infty : X \to G^\infty(\mathcal{F}, p)$ and $p_\infty : G^\infty(\mathcal{F}, p) \to Y$ such that $p = p_\infty i_\infty$.

Proposition 7.2.7 ([DS95], 7.17)**.** *In above construction, suppose that for each $i \in I$ the object A_i of $\mathcal{C}$ is sequentially small. Then the map $p_\infty : G^\infty(\mathcal{F}, p) \to Y$ has RLP with respect to each of the maps in the family $\mathcal{F}$.*

Proof. Please consider the commutative diagram,

$$\begin{array}{ccc} A_i & \xrightarrow{g} & G^\infty(\mathcal{F}, p) \\ \downarrow{\scriptstyle f_i} & & \downarrow{\scriptstyle p_\infty} \\ B_i & \xrightarrow{h} & Y \end{array} .$$

Since A_i is sequentially small, from 7.2.5, $g : A_i \to G^\infty(\mathcal{F}, p)$ factors as $g = i_{k,\infty} g'$ where $g' : A_i \to G^k(\mathcal{F}, p)$ and $i_{k,\infty} : G^k(\mathcal{F}, p) \to G^\infty(\mathcal{F}, p)$ is a natural map. Thus we get following commutative diagram,

$$\begin{array}{ccccccc} A_i & \xrightarrow{g'} & G^k(\mathcal{F}, p) & \xrightarrow{i_{k+1}} & G^{k+1}(\mathcal{F}, p) & \xrightarrow{i_{k+1,\infty}} & G^\infty(\mathcal{F}, p) \\ \downarrow{\scriptstyle f_i} & & \downarrow{\scriptstyle p_k} & & \downarrow{\scriptstyle p_{k+1}} & & \downarrow{\scriptstyle p_\infty} \\ B_i & \xrightarrow{h} & Y & \xrightarrow{=} & Y & \xrightarrow{=} & Y \end{array} .$$

Here the top composite is g. However, since (g', h) is an index in the construction of $G^{k+1}(\mathcal{F}, p)$, we get a map $B_i \to G^{k+1}(\mathcal{F}, p)$. Composing this map with the map $G^{k+1} \to G^\infty(\mathcal{F}, p)$, we get the required lifting. ■

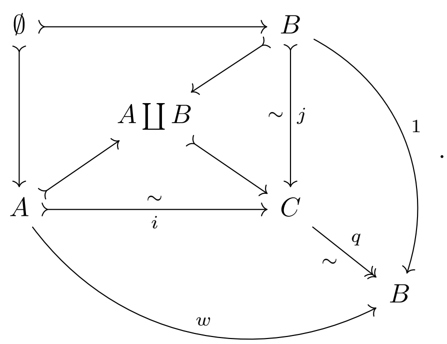
7.3 Ken Brown's Lemma

Lemma 7.3.1 ([Rie14], 11.3.14)**.** *Let $\mathcal{M}, \mathcal{N}$ be model categories and suppose $F : \mathcal{M} \to \mathcal{N}$ sends trivial cofibrations between cofibrant objects to week equivalences. Then F is homotopical on the subcategory of cofibrant objects.*

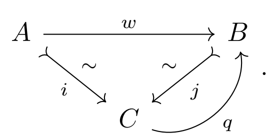
Proof. Suppose $w : A \to B$ is a weak equivalence between co-fibrant objects in $\mathcal{M}$. Consider the morphism, $w + 1 : A \coprod B \to B$, factorised as a cofibration followed by a weak fibration, q:

$$w + 1 : A \rightarrowtail C \xrightarrow[q]{\sim} B \ .$$

Since A, B are cofibrant, natural maps $A \to A \coprod B$, $B \to A \coprod B$ are co-fibrations ([Rie14], 11.1.4). Let i and j be the compositions, $i : A \to A \coprod B \to C$, $j : B \to A \coprod B \to C$, then i, j are co-fibrations. Also from 2 out of 3, i and j are weak equivalences. Now consider following diagram,



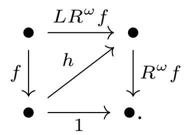
Since cofibrations are closed under taking pushouts and composition ([Rie14, 11.1.4]), i, j are cofibrations and hence trivial cofibrations. Now we get following diagram,



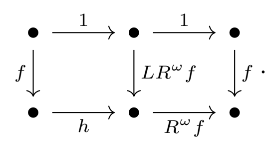
Here q is retraction of j. Now from the hypothesis, Fi, Fj are weak equivalences. By 2 out of 3 property Fq is also a weak equivalence (observe that $Fq \circ Fj = 1$). Finally $Fw = Fq \circ Fi$ is a weak equivalence again from the 2 out of 3 property. ∎

Corollary 7.3.2 ([Rie14], 12.2.4)**.** *If $\mathcal{K}$ permits the small object argument, then $^{\boxslash}(\mathcal{J}^{\boxslash})$ is the smallest class of maps containing $\mathcal{J}$ and closed under the colimits listed in the lemma [Rie14, 11.1.4]. More specifically, any map in $^{\boxslash}(\mathcal{J}^{\boxslash})$ is a retract of a transfinite composite of pushouts of coproducts of maps in $\mathcal{J}$. This shows that $^{\boxslash}(\mathcal{J}^{\boxslash})$ is the **weak saturation** of J, the smallest weakly saturated class containing $\mathcal{J}$.*

Proof. From the definition of $^{\boxslash}(\mathcal{J}^{\boxslash})$, $\mathcal{J} \subset {}^{\boxslash}(\mathcal{J}^{\boxslash})$ is immediate. Also from lemma [Rie14, 11.1.4], $^{\boxslash}(\mathcal{J}^{\boxslash})$ is closed under the colimits listed in that lemma. To show that $^{\boxslash}(\mathcal{J}^{\boxslash})$ is smallest such class we will show that any map in $^{\boxslash}(\mathcal{J}^{\boxslash})$ is a retract of a transfinite composite of pushouts of coproducts of maps in $\mathcal{J}$. Let $f \in {}^{\boxslash}(\mathcal{J}^{\boxslash})$, let $R^\omega f$ and $LR^n f$ be as in the proof of theorem [Rie14, 12.2.2]. Let $LR^\omega f$ be the canonical map $x_0 \to \text{colim}_n\, x_n$. That is LR^ω is the transfinite composite of coproducts of maps in $\mathcal{J}$. From the proof of the theorem [Rie14, 12.2.2] we get that $f = R^\omega f \circ LR^\omega f$ where $LR^\omega f \in {}^{\boxslash}(\mathcal{J}^{\boxslash})$ and $R^\omega f \in \mathcal{J}^{\boxslash}$. Since $f \in {}^{\boxslash}(\mathcal{J}^{\boxslash})$, we get following lifting diagram,



so that f is a retract of $LR^\omega f$,



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Appendix A

Monads

In a very abstract sense, a Monad is a way of encoding structure over an object in a category, such as **Set**.

A.1 Definition

Definition A.1.1 ([Rie17], 5.1.1)**.** A **monad** on a category $\mathcal{C}$ consists of

- An endofunctor, $T : \mathcal{C} \to \mathcal{C}$,
- A unit natural transformation, $\eta : \mathrm{Id}_{\mathcal{C}} \Rightarrow T$,
- A multiplication natural transformation, $\mu : T^2 \Rightarrow T$,

such that the following diagrams commute in the functor category, $\mathcal{C}^{\mathcal{C}}$:

$$\begin{array}{ccc} T^3 & \xRightarrow{T\mu} & T^2 \\ \Downarrow{\scriptstyle \mu T} & & \Downarrow{\scriptstyle \mu} \\ T^2 & \xRightarrow{\mu} & T \end{array} \qquad \begin{array}{ccccc} T & \xRightarrow{\eta T} & T^2 & \xLeftarrow{T\eta} & T \\ & {\scriptstyle \mathrm{Id}_T}\searrow & \Downarrow{\scriptstyle \mu} & \swarrow{\scriptstyle \mathrm{Id}_T} & \\ & & T & & \end{array} . \tag{A.1.1}$$

A variety of intuitions are available:

- "A monad is just a monoid in the category of endofunctors, what's the problem?" – Philip Wadler (apocryphal)
- "A monad is the 'shadow' cast by an adjunction on the category on the appearing on the as the codomain of the right adjoint." [Rie17]

The most intuitive example that I found online was the monoid example from the Catsters video series on monads [Che14]:

Example A.1.2 (The Monad for Monoids)**.** We set $\mathcal{C} = \mathbf{Set}$, and we need to describe three pieces of data (that satisfy the commutativity laws).

- T takes a set A to the set of finite lists of elements in A (we will denote a list with brackets instead of braces).
- To define $\eta : \mathrm{Id}_{\mathbf{Set}} \Rightarrow T$, we need to create a set of lists out of a simple set. An "obvious" way to do this that preserves all of the data is to directly convert elements into lists, and indeed, this is what we do:

$$\eta_A(a) = [a].$$

- To define $\mu : T^2 \Rightarrow T$, we note that the set T^2A is the set of lists of lists of elements of A, which is a bit of a mouthful. We are hoping to land in TA, which is simply the set of (finite) lists of elements of A. A natural guess of how to do this is concatenation: a list of lists goes to the list made by concatenating all of its lists. For example,
$$\mu([[a,b],[c],[d,e]]) = [a,b,c,d,e].$$

Now we need to check that we actually have defined a Monad (since all we have done so far is make Category-theoretical guesses on what everything *has* to be for the math to work out).

First we check the square: An element in $T^3(A)$ is, in words, a list of lists of lists. That is, three levels of stacked brackets, such as:
$$x = [\,[\,[a,b],[c]\,],[\,[d],[e,f]\,]\,].$$
Taking the upper right path along the square (A.1.1) concatenates the the interior brackets together, while taking the lower left path concatenates the out brackets first. The last path for each direction does the same thing, since there is no whiskering.

In terms of this example, we have the square:

$$\begin{array}{ccc}
[\,[\,[a,b],[c]\,],[\,[d],[e,f]\,]\,] & \xrightarrow{T\mu_x} & [\,[a,b,c],[d,e,f]\,] \\
\downarrow{\scriptstyle \mu_{Tx}} & & \downarrow{\scriptstyle \mu} \\
[\,[a,b],[c],[d],[e,f]\,] & \xrightarrow{\mu} & [a,b,c,d,e,f]
\end{array}$$

In this example, the square certainly commutes, but it also does generally, since what we are ascertaining is that it doesn't matter which order we get rid of the (interior) brackets in, which is generally true.

For the other triangle(s), we need to check that the composition $\mu \circ \eta T$ is the identity on T. Recall that η simply adds brackets (i.e. makes a list of) every element in the set being considered. However, there are multiple ways to do this. We can either place an extra layer of brackets around each of the interior lists ($T\eta$) or place a layer of brackets around the big list (ηT). Either way, under concatenation, these extra brackets will vanish, leaving us with the same list that we started with. To make this more explicit, we consider another example,
$$y = [a,b,c].$$

$$\begin{array}{ccc}
[a,b,c] & \xrightarrow{\eta_{Ty}} & [[a,b,c]] \\
& {\scriptstyle \mathrm{Id}_{Ty}}\searrow & \downarrow{\scriptstyle \mu} \\
& & [a,b,c]
\end{array}
\qquad\qquad
\begin{array}{ccc}
[[a],[b],[c]] & \xleftarrow{T\eta_y} & [a,b,c] \\
\downarrow{\scriptstyle \mu} & \swarrow{\scriptstyle \mathrm{Id}_{Ty}} & \\
[a,b,c] & &
\end{array}$$

Important Point: None of these natural transformations can reduce down to the level of the original category. We will always be living in the image of T. So, although we have constructed a monad for monoids (or rather, we claim to have), we don't actually have a single monoid yet. The only reason that we might consider this monad to be reminiscent of a monoid is that words are sort of like products of elements of the monoid. But to actually find a value for the word *within* a monoid, we need a map to the underlying set of the monoid.

That is, if A is the underlying set of a monoid, then for $a,b,c \in A$, we can form the word, $[a,b,c] \in TA$. And further, we know that the way we stack lists doesn't matter, so $[[a,b],c]$ and $[a,[b,c]] \in T^2A$ both map to $[a,b,c] \in TA$, which is reminiscent of associativity (and will be associativity in the monoid we are going to construct). However, we don't know what element of A $[a,b,c]$ corresponds to. Thus, to encode any structure more specific than that of a general monoid, we need to provide a morphism, $TA \to A$, that tells us what the value of $[a,b,c]$ actually is *in the monoid*.

The underlying set A (i.e. an object of the category we are considering) and the morphism $TA \to A$ together form an algebra over the monad (which in this case is a monoid). In a sense, the monad provides as much information about the monoid as it can generally, and the algebra fills in the gaps of information that can't be presented generally. We will now be more precise:

Definition A.1.3. An **algebra** over a monad, $T : \mathcal{C} \to \mathcal{C}$ is an object $A \in \mathcal{C}$ and morphism $\mathrm{ev} : TA \to A$ satisfying the following commutativity conditions:

$$\begin{array}{ccc} T^2A & \xrightarrow{\mu_A} & TA \\ \downarrow{\scriptstyle T(\mathrm{ev})} & & \downarrow{\scriptstyle \mathrm{ev}} \\ TA & \xrightarrow{\mathrm{ev}} & A. \end{array} \qquad \begin{array}{ccc} A & \xrightarrow{\eta_A} & TA \\ & {\scriptstyle \mathrm{Id}_A}\searrow & \downarrow{\scriptstyle \mathrm{ev}} \\ & & A \end{array}$$

In the case of the monad for monoids, an $A \in \mathcal{C}$ is a set, and the map $\mathrm{ev} : TA \to A$ gives the evaluation of a list/word in TA. This also makes clear what the identity element of the monoid is: the empty string, $[\,] \in TA$ must have an image $\mathrm{ev}([\,]) \in A$. The commutativity conditions ascertain that $\mathrm{ev}([\,])$ is the identity element of the monoid. It's worth noting that while the identity element is a specific element of the algebra's underlying set, it's existence is enforced entirely by the monad, and it's in this way that the monad for monoids encodes the existence of an identity element.

List of Exercises

References

[Che14] Eugenia Cheng. Monads. https://www.youtube.com/playlist?list=PL0E91279846EC843E, 2014. Accessed: 20-08-04.

[DS95] W. G. Dwyer and J. Spalinski. *Homotopy theories and model categories*. Elsevier Science, 1995.

[Dug08] Daniel Dugger. A primer on homotopy colimits. *preprint*, 2008.

[Dwy05] William G Dwyer. *Homotopy limit functors on model categories and homotopical categories*. American Mathematical Soc., 2005.

[Mal07] Georges Maltsiniotis. Le théorème de quillen, d'adjonction des foncteurs dérivés, revisité. *ScienceDirect*, 2007.

[ML13] Saunders Mac Lane. *Categories for the working mathematician*, volume 5. Springer Science & Business Media, 2013.

[Ric20] B. Richter. *From Categories to Homotopy Theory*. Cambridge Studies in Advanced Mathematics. Cambridge University Press, 2020.

[Rie14] Emily Riehl. *Categorical homotopy theory*, volume 24. Cambridge University Press, 2014. http://www.math.jhu.edu/~eriehl/cathtpy.pdf.

[Rie17] Emily Riehl. *Category theory in context*. Courier Dover Publications, 2017. http://www.math.jhu.edu/~eriehl/context.pdf.

[Wof16] Eric Wofsey. Enriching categories of simplicial objects. Mathematics Stack Exchange, 2016. https://math.stackexchange.com/q/1757704 (version: 2016-04-25).