Recursion Relationship for First Return Times

The probability of first returning to state \( i \) at time \( n \) is denoted by

\[
f_{ii}^{(n)} = P(X_n = i, X_k \neq i, k = 1, \ldots, n - 1 \mid X_0 = i),
\]

with \( f_{ii}^{(0)} = 0 \). The (not necessarily first) \( n \)-step return probabilities \( P_{ii}^{(n)} \) can be expressed as a function of the \( n \)-step first return probabilities \( f_{ii}^{(k)}, k = 0, \ldots, n \):

\[
P_{ii}^{(n)} = \sum_{\text{all paths } \{X_l\}_{l=0}^n \text{ with }} \sum_{X_0 = i \text{ and } X_n = i} P(\{X_l\}_{l=0}^n) \tag{1}
\]

\[
= \sum_{k=0}^n P(\text{first return to } i \text{ in } k \text{ steps and then end up at } i \text{ after } n \text{ total steps}) \tag{2}
\]

\[
= \sum_{k=0}^n P(\text{first return to } i \text{ in } k \text{ steps}) P(X_n = i \mid X_k = i)
\]

\[
= \sum_{k=0}^n f_{ii}^{(k)} P(n-k).
\]

The key is seeing that the paths described in (1) can be decomposed into the paths described in (2).

Example

Consider a Markov chain on three states \( \{0, 1, 2\} \) with the transitions shown in the diagram:

![Diagram](image)

In other words, transitions from any state \( i \) are allowed to any state \( j \) at each step. To calculate \( P_{00}^{(3)} \) we would need to sum over all paths that start at 0 and end up at 0 after three steps. The possible paths are:
The first box corresponds to paths that have first return times at \( n = 1 \); the second box corresponds to paths with first return times at \( n = 2 \); the third box has paths with first return times at \( n = 3 \). This shows that we can partition all paths with \( X_0 = 0 \) and \( X_3 = 0 \) into groups that have first return times at \( k = 1 \), \( k = 2 \) and \( k = 3 \).