Recurrence Properties for Random Walks
with a Partially Reflecting Boundary

Positive Recurrent Case

Consider a random walk on the integers \(\{0, 1, 2, \ldots\}\), where \(P_{00} = 1 - p\), \(P_{i,i-1} = 1 - p\) for \(i = 1, 2, \ldots\), and \(P_{i,i+1} = p\) for all \(i\). When \(p < 1/2\), one can show that the chain is positive recurrent. A sample path is shown in the plot below for \(p = 0.4\). The chain was started at \(X_1 = 10\). Note how quickly the chain moves to state 0 and then how often it returns in the future.

The limiting distribution for this chain for a general \(p\) is

\[
\pi_j = \left(\frac{1 - 2p}{1 - p}\right) \left(\frac{p}{1 - p}\right)^j, \quad j = 0, 1, \ldots,
\]

and so \(\pi_0 = 1/3\) when \(p = 0.4\). This means the chain will return to state 0 on average every three steps. The plot below shows a Monte Carlo estimate of \(m_0\) based on about 35,000 independent samples from the distribution of first return times.
**Null Recurrent Case**

The chain is null recurrent if $p = 1/2$. This means that the chain will return to state 0 with probability 1, however the expected time until it returns is infinite. The plot below shows a sample path of length 50,000 from this null recurrent chain. Even though the chain has wandered far away from 0 after 50,000 iterations, it will return with probability 1.

![Sample Path of Null Recurrent Chain](image)

Because the chain is null recurrent, $m_0 = \infty$. If we did not realize this and tried to estimate this quantity via Monte Carlo integration, the estimate might look something like the following plot.

![Monte Carlo Estimate](image)

This plot is based on about 2000 independent samples from the distribution of first return times. The behavior in the plot is often a good indication that the quantity you are trying to estimate is not finite.
Transient Case

When \( p > 1/2 \) the chain is transient. This means the chain will return to state 0 a finite number of times with probability one and then will never return. The following figure displays a sample path when \( p = 0.6 \).

\[ \begin{array}{c}
p = 0.6 \\
\end{array} \]

![Sample path](image)

In this case, the chain was started at state zero and returned to zero exactly four times. It can be shown that the probability the process ever returns to zero given that it starts at zero is

\[ f_{00} = P(X_n = 0 \text{ for some } n > 0 \mid X_0 = 0) = 2(1 - p) \]

for the transient \((p > 1/2)\) case. This means that \( M \), the random variable representing the number of times the Markov chain returns to state zero, has the Geometric\((2p-1)\) distribution. When \( p = 0.6 \) the probability mass function is

\[ p_M(k) = 0.2 \cdot (1 - 0.2)^k, \quad k = 0, 1, \ldots \]

and the expected number of returns is \( f_{00}/(1 - f_{00}) = 4 \). For the simulation shown above, it appears as though the process will never return to state 0 since it has wandered so far away, however this is not guaranteed. When \( p = 0.6 \), \( P(M > 4) = 0.32768 \). We have a bit more information about the particular process that was simulated above: we know that it is currently at state \( X_{1000} = 158 \). So the probability that it will return to zero in the future is

\[ P(X_n = 0 \text{ for some } n > 0 \mid X_0 = 158) = \sum_{n=1}^{\infty} P_{158,0}^{(n)} = \sum_{n=158}^{\infty} P_{158,0}^{(n)}. \]

This would take some work to calculate; the result should be extremely small.
Another Look

You might think that the reason the sample paths look as different as they do is because the three values of $p$ are “far” apart. The following plot shows three sample paths for $p \in \{0.49, 0.50, 0.51\}$. You might not expect these paths to look remarkably different, however they exhibit the same behavior seen above.