Statistics 773: Homework #3
Due 11/9/2005

**Question 1** [Based on RC 2.23]
This problem involves generating from a Beta($\alpha, \beta$) distribution.

(a) Show that, if $Y_1 \sim \text{Gamma}(\alpha, 1)$ and $Y_2 \sim \text{Gamma}(\beta, 1)$, then
\[ X = \frac{Y_1}{Y_1 + Y_2} \sim \text{Beta}(\alpha, \beta). \]

(b) Use (a) to take 1000 draws from a Beta(2,6) distribution using only draws from a Unif(0,1) distribution (show a histogram and report the mean and variance of the draws). [Use the fact that if $Z_i \sim \text{Exp}(1)$, then $\sum_{i=1}^{\nu} Z_i \sim \text{Gamma}(\nu, 1)$.] Make a quantile-quantile (Q-Q) plot of your 1000 draws by using a beta r.v. generator in a computing package (e.g. `rbeta()` in R).

(c) Implement an Accept-Reject method for drawing from a Beta($\alpha, \beta$) distribution using a Unif(0,1) instrumental distribution. Comment on any restrictions this places on possible values of $\alpha$ and $\beta$. Find the optimal bound, $M_{\text{opt}}$, and the acceptance probability. Use this method to take 1000 draws from a Beta(2,6) distribution, make a Q-Q plot, and report how many total draws were needed to obtain your 1000.

(d) Investigate the efficiency of this proposal distribution by computing the acceptance probability over a grid of points for $\alpha$ and $\beta$ on the rectangle $[1, 10] \times [1, 10]$. Make a 3-d plot with the acceptance probability on the “z”-axis (try using the `wireframe` function from the `lattice` library in R). Comment on why the method is more efficient for particular regions of the rectangle.

**Question 2** [RC 2.30(a)]
For the Accept-Reject algorithm, with $f$ and $g$ properly normalized, when the bound $M$ is too tight (i.e., when $f(x) > Mg(x)$ on a non-negligible part of the support of $f$), show that the Accept-Reject algorithm does not produce a generation from $f$. Give the resulting distribution.

**Question 3** [RC 2.7(a)]
Prove carefully that the Box-Muller algorithm returns independent standard normal random variates. The Box-Muller algorithm takes $U_1$, $U_2$ as independent Uniform(0,1) random variables and then makes the transformation:
\[ X_1 = \sqrt{-2\log(U_1)} \cos(2\pi U_2), \quad X_2 = \sqrt{-2\log(U_1)} \sin(2\pi U_2). \]