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ARTICLES

Astrophysical applications of quantum corrections to the equation of state of a plasma

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The quantum electrodynamic correction to the equation of state of a plasma at finite temperature is applied to the areas of solar physics and cosmology. A previously neglected, purely quantum term in the correction is found to change the equation of state in the solar core by -0.37%, which is roughly estimated to decrease the calculated high energy neutrino flux by about 2.2%. We also show that a previous calculation of the effect of this correction on big bang nucleosynthesis is incomplete, and we estimate the correction to the primordial helium abundance Y to be $\Delta Y = 1.4\times 10^{-4}$. A physical explanation for the correction is found in terms of corrections to the dispersion relation of the electron, positron, and photon.

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I. INTRODUCTION

Although there is a well-developed theory regarding the electromagnetic corrections to the equation of state of a plasma of charged particles [1–5], the theory has not been consistently applied to astrophysical environments. For example, many astrophysical models, such as solar models [6], use the classical approach to calculating the electromagnetic correction to the pressure, known as the Debye-Hückel theory, which yields a relative correction of order $e^3 n_e^{1/2}/T^{3/2}$, where n_e is the electron number density [1]. This theory is valid as long as this correction remains small, or as long as $n_e^{1/2} \ll T^{3/2}/e^3$. This constraint, however, is only *marginally* held in many of these astrophysical models, which often have higher densities.

In order to calculate the correction to the pressure at higher densities, one must use quantum theory. A direct calculation of the correction using well-established finite temperature quantum electrodynamics (QED) [2–4] produces an expansion series in terms of the parameters $(e,\hbar,c,n_e,T,m)$. Along with the classical e^3 term, this expansion series includes a purely quantum mechanical term proportional to e^2. For example, in the classical regime, the QED expansion produces a term proportional to $e^2 n_e \hbar^2/mT^2$. Naturally, the size of the e^2 term depends on the temperature and density, but in many astrophysical environments this e^2 term can be comparable to, and even greater than, the classical e^3 term.

In this paper we will look at a few examples where the e^2 term can be an important correction to the equation of state, and although the corrections are small in absolute terms, they can still produce observable differences.

It should be noted that in this paper we will be concerned only with the regime where the electromagnetic interactions can be treated as *perturbations* to the ideal plasma behavior. Otherwise the QED expansion series would not be useful, and another approach would be needed to find the equation of state [5]. The constraints on the temperatures and densities of this "weakly interacting" plasma regime are further discussed at the end of Sec. II.

The outline of this paper is as follows. In Sec. II we examine the general and technical properties of the QED correction to the equation of state over a wide range of temperatures and densities. The reader more interested in specific results for solar physics or cosmology can go directly to Secs. III or IV, where we have estimated the effects of the correction in these environments. Finally in Sec. V, we will examine the finite temperature dispersion relation of the electrons and photons because the dispersion relation is not only necessary for determining the weak reaction rates, but it is also intimately related to the QED pressure correction and gives physical insight into this correction.

II. THE QED CORRECTION

When applying the e^2 term to solar physics and cosmology, we will find that the contribution from the nuclei is negligible. Therefore we will concentrate only on the $e^{\pm}$-photon plasma.

The first corrections to the pressure (or the free energy) of the $e^{\pm}$-photon plasma due to electromagnetic coupling are represented as the Feynman diagrams shown in Fig. 1. Many authors have calculated these graphs [2–4], and we have included the general result for the e^2 correction

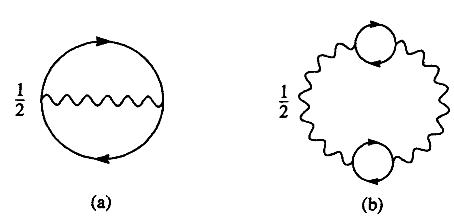

FIG. 1. The Feynman diagrams for the pressure correction. (a) represents the e^2 correction, and (b) is just one of an infinite sum of loop diagrams which represent the e^3 term [4]. The coefficients in front are symmetry factors.

in Appendix A. For astrophysics, there are two limits that are of particular interest, namely $T \gg mc^2$ and $T \ll mc^2$. For $T \gg mc^2$ or $\mu \gg m$, the correction to the pressure (of an ideal plasma) $P_{\rm int}$ takes the simple form

$$P_{\rm int}(\mu,T) = -\frac{e^2}{288\hbar^4 c^4}\left[5T^4 + \frac{18}{\pi^2}\mu^2T^2 + \frac{9}{\pi^4}\mu^4\right] + \frac{e^3 T}{12\pi h^{9/2}c^{9/2}}\left[\frac{T^2}{3}+\frac{\mu^2}{\pi^2}\right]^{3/2} + O(e^4)\,, \tag{1}$$

and in the nonrelativistic, nondegenerate limit $\alpha^2 mc^2 \ll T \ll mc^2 - \mu \ll mc^2$,

$$P_{\rm int}(\mu,T) = \frac{e^2 m^2 T^2 e^{2(\mu-m)/T}}{16\pi^3\hbar^4} + \frac{e^3}{12\pi^3\hbar^{9/2}}\left[\frac{m^9T^7}{8\pi}\right]^{1/4} e^{(3/2)(\mu-m)/T} + O\left[\frac{e^4 n_0^2\hbar}{m^{1/2}T^{3/2}}\right]. \tag{2}$$

Here μ is the electron chemical potential, m is the electron mass, $e^2/4\pi\hbar c = \alpha$, the fine structure constant, we define $n_0 \equiv 2(mT/2\pi\hbar^2)^{3/2}e^{(\mu-m)/T}$, and we will set the Boltzman constant $k=1$. The lower limit on the temperature and the higher order correction in (2) will be discussed at the end of this section. We can use the definition $n_e = \partial P/\partial\mu$ and the result of (2) to get an equation of state of an electron plasma in the classical regime:

$$P_e(n_e,T) \approx n_e T\left[1 - \frac{e^2 n_e \hbar^2}{8mT^2} - \frac{e^2 n_e^{1/2}}{12\pi T^{3/2}}\right]. \tag{3}$$

The approximation sign in (3) is there to remind that we are neglecting the higher-order terms. Here, one can see explicitly that the e^2 term is due to quantum-mechanical effects, since it includes $\hbar$, and the e^3 term is the familiar Debye-Hückel term, which can also be calculated classically. Both of the corrections in (3) are *nonrelativistic* since there is no factor of c in them. Note the interesting behavior of the e^2 term in the two limits of equations (1) and (2): the e^2 term *changes sign,* depending on the temperature and density. This change of sign of the QED correction will also be discussed further in Sec. V.

As shown in Eqs. (2) and (3), there are two different ways to present the results of the correction. The first way is to keep the *chemical-potential constant* when turning on the interaction. Then the total pressure is $P(\mu,T) = P_0(\mu,T) + P_{\rm int}(\mu,T)$. Figure 2 graphically shows a summary of the results of the (e^2 part) of the correction $P_{\rm int}(\mu,T)$, which can be calculated generally by using the formula in Appendix A. Astrophysical environments can cover a wide range of this parameter space. Notice that as the density increases, there is a point at which the correction flips from positive to negative (at low T), as seen from curve (d) to (e). This change in sign at high densities is better shown in Fig. 3, which shows the correction as a function of chemical potential at zero temperature. The correction changes sign at a chemical

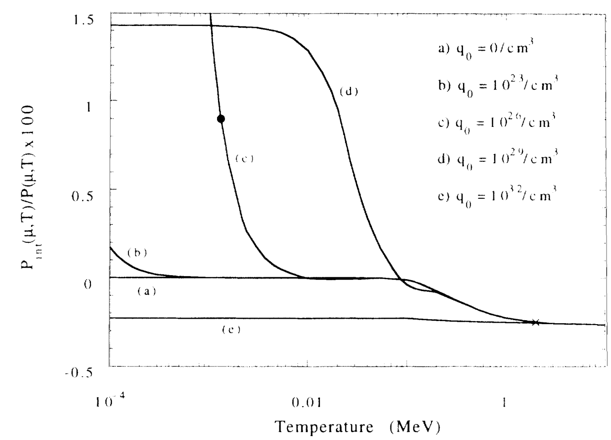

FIG. 2. The percent e^2 correction to the pressure $P_{\rm int}(\mu,T)$, where the total pressure is $P(\mu,T) \equiv P_0(\mu,T) + P_{\rm int}(\mu,T)$, for lines of constant (ideal gas) electron charge density $q_0(\mu,T) = n_0^{e^-}(\mu,T) - n_0^{e^+}(\mu,T)$. Using lines of constant density with the full interacting gas definition of density $n \equiv \partial P/\partial\mu$ would only negligibly change the lines on this graph. $P(\mu,T)$ is the total pressure of the $e^{\pm}$-photon plasma. The "$\times$" marks the conditions when neutron (and neutrino) decoupling occurs in the big bang, and the dot is at the conditions for the solar core.

potential $\mu_{\rm crit} \approx 2.7 m_e c^2$ which corresponds to a density $n_{\rm crit} \approx 9.47\times10^{30}/{\rm cm}^3$. At low densities and temperatures the correction can become quite large.

The second way to present the results of the correction, is to keep the *density constant* when turning on the interaction. This is a more practical way in the sense that the total pressure is in the form of an equation of state: $P(n_e,T) = P_0(n_e,T) + P_{\rm int}(n_e,T)$, where $n_e = \partial P/\partial\mu$ is the corrected density. Note that in this notation, $P_{\rm int}(n_e,T)$ and $P_{\rm int}(\mu,T)$ are *two different functions.* To first order in the correction, they are related by the equation $P_{\rm int}(n_e,T) = P_{\rm int}(\mu_0,T) - (n_0/\partial n_0/\partial\mu)\partial P_{\rm int}(\mu_0,T)/\partial\mu$, where $n_0 = \partial P_0(\mu_0,T)/\partial\mu$ and μ_0 is the chemical potential of the noninteracting gas. This relation now al-

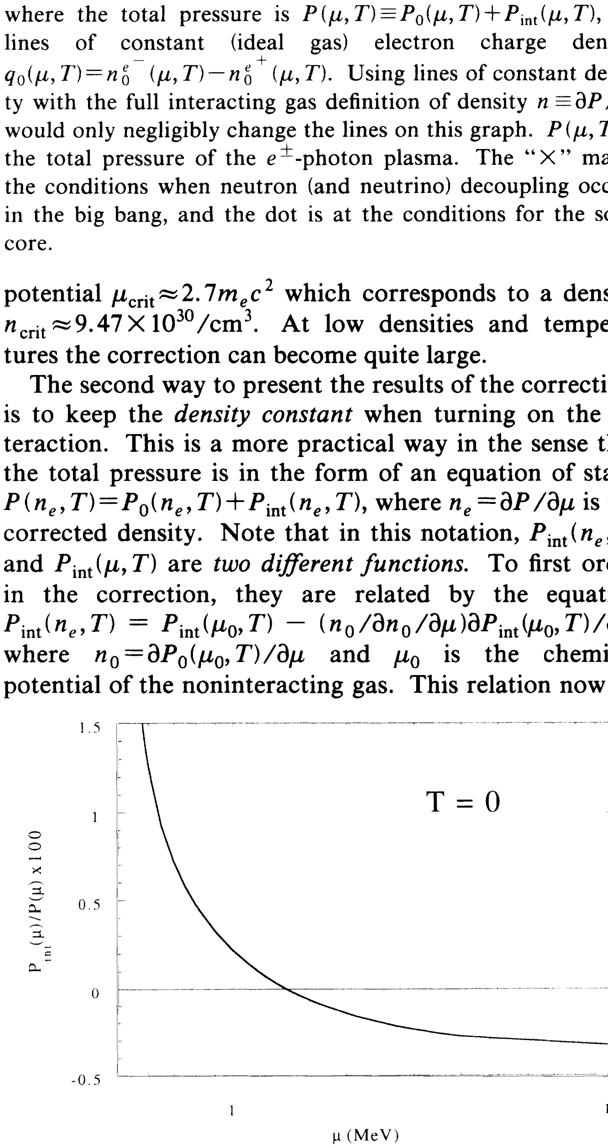

FIG. 3. The percent e^2 correction to the pressure as a function of electron chemical potential at $T=0$. Notice that the correction changes sign at a density of about 10^{31} cm^{-2}, and at low densities, the correction can be quite large because the potential energy per particle is becoming large compared to the kinetic energy.

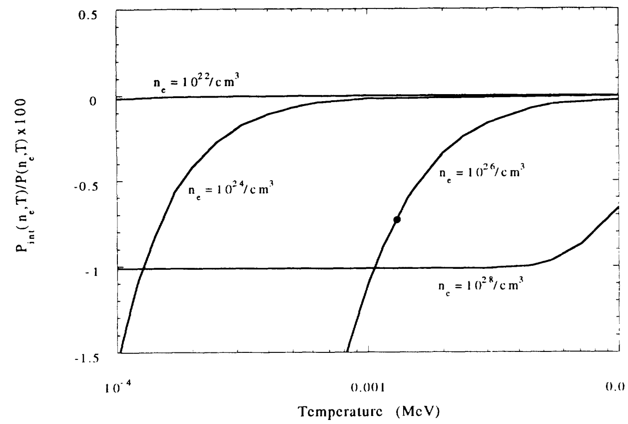

FIG. 4. The percent e^2 correction to the pressure $P_{\rm int}(n_e,T)$, where the total pressure is $P(n_e,T)\equiv P_0(n_e,T)+P_{\rm int}(n_e,T)$ (see text), for lines of constant electron density n_e. This graph is useful for finding the e^2 correction to the equation of state of a plasma at nonrelativistic temperatures. The dot is at the conditions for the solar core.

lows us to calculate $P_{\rm int}(n_e,T)$ using the formula in Appendix A. Figure 4 is a graph of $P_{\rm int}(n_e,T)$, and we will use this graph when applying the correction to the solar core.

There is one more question about this calculation: What about higher order terms? This is a very important question because in the solar core, for example, the e^3 term is larger than the e^2 term! How can we be confident that the higher-order terms are not larger still? The following arguments will show that as long as the temperature or density are sufficiently high, all of the higher-order terms are expected to be at most only $\sim 15\%$ as large as the e^2 or e^3 terms.

For the simple case of high temperature, $T \gg mc^2$, the nth-order correction will be of the form $\sqrt{\alpha^n}T^4$ to all orders, so it is easy to see that higher-order terms are small. The same argument goes for $\mu \gg mc^2$. For $T=0$, an analysis of the correction shows that we must constrain $\alpha mc/p_I \ll 1$ [where p_F is the Fermi momentum, $p_F c=(\mu^2-m^2c^4)^{1/2}$], in order for all terms in the expansion to be small. This limit ensures that the average kinetic energy is much greater than the average potential energy, so that electromagnetic interactions are small perturbations. The correction in Fig. 4 blows up because the potential energy is becoming large compared to the kinetic energy per particle. As long as $\alpha mc/p_F \ll 1$, or, in terms of density, as long as $n \gg (\alpha mc/\hbar)^3$, all higher-order terms are negligible. This corresponds to a minimum density $n_{\rm min} \gg 10^{25}/{\rm cm}^3$.

Finally, let us look at the regime $mc^2-\mu \sim T \ll mc^2$, which is nonrelativistic, slightly degenerate, and relevant to the solar core (this argument is also valid for $T \ll m-\mu$). All of the terms in the expansion of the correction must meet certain requirements. First, they must all have powers of e and n in the numerator, since each bubble in a diagram corresponds to a factor of e^2 and n. And since we are in a nonrelativistic regime, none of the terms can have a factor of c in them. Using these requirements, and by reformulating the expansion in terms of dimensionless combinations of the possible parameters $(e,\hbar,c,n_0,T,m)$, one finds that all terms must be a function of the general form $(e^2m^{1/2}/4\pi\hbar T^{1/2})^a(n_0\hbar^3/m^{3/2}T^{3/2})^b$ where a and b are integers and $a \ge b > 0$ [7]. Strictly speaking, there are also terms that include the photon density $n_{\rm ph}$, but these terms have factors of $n_{\rm ph}\sim T^3/m^3c^6$ in them and so are neglected. An analysis of this form shows that the next largest term after the $e^2n_0\hbar^2/4\pi mT^2$ term (in addition to the e^2 term) is $e^4n_0\hbar/(16\pi^2m^{1/2}T^{5/2})$ which is smaller by a factor of $(e^2\hbar/4\pi)(m/T)^{1/2}$. So as long as $T \gg \alpha^2mc^2$, all higher-order terms will be negligible. This constraint is similar to the $T=0$ constraint above in that it ensures that the average kinetic energy per particle is much greater than the average potential energy, and since α^2mc^2 is the Rydberg energy, it ensures that the plasma is ionized. For conditions inside the solar core (see Sec. III), $(e^2\hbar/4\pi)(m/T)^{1/2}\approx 15\%$, so the largest higher-order corrections are $\approx 15\%$ as large as the e^2 or e^3 terms. Of course, this analysis does not include the geometrical factors, which tend to make the higher-order terms even smaller.

III. APPLICATION TO SOLAR PHYSICS

Perhaps the most exciting application of this correction is to the solar interior. The solar models now used [6] all include the classically calculated Debye-Hückel (DH) term, but they do not include the e^2 correction in their equation of state. In this section we will estimate the effect of the e^2 correction on the solar-neutrino flux.

Using typical values for the solar core of $T=1300$ eV and $n_e=10^{26}/{\rm cm}^3$, and using Fig. 3 [Eq. (3) is a rough approximation], we find a negative correction to the electron pressure (here we are calculating the e^2 term only):

$$\frac{P_e(n_e,T)-P_0(n_e,T)}{P_e} \simeq -0.0073 \ . \tag{4}$$

For simplicity, let us assume that the solar core is only made up of protons and electrons (even if the core were 50% helium by number, this would only change the final answer by 25%). Then the e^2 term change to the total pressure is $\Delta P_{\rm tot}^{e^2 \rm term}/P_{\rm tot}=-0.0037$, because the proton contribution to the change in the pressure is smaller than the electron contribution by a factor of m_e/m_p, and can be neglected.

We can estimate how this correction affects the solar-neutrino flux by comparing our change in the equation of state with solar model data in Ref. [6] (Table III), which calculates neutrino fluxes with and without the DH effect included in the equation of state [8]. This data shows that the DH effect decreases the neutrino flux by 0.5 solar neutrino units (SNU) (6%) for the chlorine experiment, and 3 SNU (2%) for the gallium experiment. Since the DH effect decreases the pressure of the solar interior by about 1% [6], and the e^2 term decreases the pressure by 0.37%, we estimate the change in the neutrino flux to be

$$\frac{\Delta\text{flux}}{\text{flux}} \approx -0.19\ \text{SNU} \approx -2.20002 \text{ for chlorine} \ , \tag{5}$$

$$\frac{\Delta\text{flux}}{\text{flux}} \approx -1.1\ \text{SNU} \approx -0.7\% \text{ for gallium} \ . \tag{6}$$

Let us restate that these estimates include only the QED changes to the equation of state for the e^2 term. There is another QED correction to the classical DH term itself, and the exact calculation [4] for an electron gas shows that the DH term is about 15% *lower* [9] than classically calculated, and this reduction translates to an additional $\Delta P/P \approx 0.2\%$ or Δflux/flux≈ 0.1 SNU$\approx 1\%$ for chlorine and Δflux/flux≈ 0.5 SNU$\approx 0.4\%$ for gallium. These corrections are to be added on to (5) and (6).

There is one last issue concerning corrections to the solar-neutrino flux: Along with corrections to the equation of state of the solar core, there are also QED corrections to the *energy spectrum* of the electrons. These corrections to the electron-energy spectrum will affect the weak reaction rates in the solar core, and consequently the neutrino flux. But we will see that the extra correction to the flux, due to changes in the electron-energy spectrum, is small.

To show this, we will use the QED dispersion relation of the electrons to find the change in reaction rates. We will assume that the QED electron dispersion relation takes the form $E^2=p^2c^2+m_0^2c^4+\delta m^2c^4$, where δm^2 is the QED correction to the mass (see Sec. V). All weak reaction rates R involving electrons (such as $e^-+\mathrm{Be}\rightarrow\mathrm{Li}+\nu$) are roughly proportional to $R\sim\int_0^\infty d^3p\,\sigma(E)f_e(E)$, where $\sigma(E)\sim(m/m_0)^5(a+bE/m+cE^2/m^2)$ is the two particle cross section, $f_e(E)$ is the electron occupation number, and a, b, and c are constants. It is helpful to reformulate R in terms of statistical averages of powers of the energy

$$R\sim\int d^3p\,(m/m_0)^5(a+bE/m+cE^2/m^2)f_e(E)$$
$$=(m/m_0)^5(a+b\overline{E}/m+c\overline{E^2}/m^2)n_e\ ,$$

where $\overline{E^n}\equiv(\int d^3pE^nf(E))/n_e$. If we now use the QED dispersion relation in place of the vacuum dispersion relation in the expression for R, we find the correction to the reaction rates

$$\frac{\delta R}{R}\sim\frac{5}{2}\frac{\delta m^2}{m^2}+\frac{\delta n_e}{n_e}\ .$$

The first term in this correction to the reaction rates is due to the correction to the energy spectrum of the electrons, while the second term is due to the correction in the number density. We can compare these two terms by numerical estimation. By using (15) in Appendix B and by using (4) and approximating $\delta n_e/n_e\sim\delta P_e/P_e\approx 0.01$ for the solar core, we find that $\delta n_e/n_e\gg\delta m^2/m^2$, for conditions inside the solar core. That is to say, the greatest QED correction to reaction rates is due to the change in the number density of the electrons in the plasma, and the correction due to the change in the energy spectrum is negligible. The solar code used to obtain (5) and (6) takes into account the changes in reaction rates due to corrections in density, but it does not take into account the changes in reaction rates due to corrections in the energy spectrum. Since the changes in the reaction rates due to corrections in the energy spectrum are negligible, to a good approximation (5) and (6) accurately include the corrections to the reaction rates.

Although the effect of the QED correction on the expected neutrino flux is small, this estimate shows that the order of magnitude of this effect is large enough to be included in solar codes, where one can determine the full extent of this effect.

IV. APPLICATION TO COSMOLOGY

The production of elements, such as helium, in the big bang is very sensitive to the conditions of the Universe at the time of big bang nucleosynthesis (BBN). Let us first discuss how finite temperature QED changes these conditions, and then we will estimate how these changes affect helium production in BBN.

Finite temperature QED changes the conditions in the early Universe ($T\gg mc^2,\mu$) in three ways. First, one can use Eq. (1) to find that the QED correction reduces the energy density and pressure in the early Universe at a given temperature. This can be translated into a decrease in the number of degrees of freedom of the plasma:

$$\rho_{\rm tot}=(N+\delta N_{\rm QED})\frac{\pi^2}{15\hbar^3c^3}T^4\ . \tag{7}$$

Equation (1) is a good approximation, but the exact e^2 term calculation (see Appendix A) and the inclusion of the e^3 term in (1) yields $\delta N\simeq -0.006$ for $T=1$–2 MeV. At this temperature the number of degrees of freedom $N=\frac{43}{8}$ which includes photons, $e^\pm$, and three types of neutrinos. Therefore,

$$\frac{\delta N}{N}\approx -0.0011\ . \tag{8}$$

Notice that this change in the number of degrees of freedom could also be interpreted as changing the fractional number of dark matter species allowed at the time of BBN. If we assume they are neutrinos, then the change in the maximum allowable number of neutrino species is $\Delta N_\nu=+0.01$.

Second, there is a QED correction to the dispersion relation of the electrons. This is discussed later in this section and particularly in Sec. V.

Third, as a consequence of the correction to the energy density of the $e^\pm$-photon plasma, there are two modifications to the well-known [11] relation between the neutrino temperature and the photon temperature $T_\nu=(\frac{4}{11})^{1/3}T_\gamma[\mathcal{S}(m_e/T_\gamma)]^{1/3}$. First, there is a change to the coefficient $(\frac{4}{11})^{1/3}$ because the energy density before the $e^\pm$ annihilations is decreased by QED corrections (afterwards, the correction is negligible because $n_e\approx 0$). That is to say, the photon temperature will be lower than in the standard relation because there is less total energy in the plasma to transfer to the photons. Second, we must make an addition to $\mathcal{S}(m_e/T)$ which takes into account the temperature dependent part of the QED correction. The new relation is

$$T_\nu=\left[\frac{1}{\frac{11}{4}+\delta N}\right]^{1/2}T_\gamma\left[\mathcal{S}\left[\frac{m_e}{T_\gamma}\right]+\mathcal{S}'\left[\frac{m_e}{T_\gamma}\right]\right]^{1/3}\ , \tag{9}$$

where

$$\mathcal{S}'(m_e/T_\gamma)=\frac{15\hbar^3c^3}{\pi^2T^4}[\rho_{\rm int}(T_\gamma)+P_{\rm int}(T_\gamma)]\ ,$$

and $\rho_{\rm int}, P_{\rm int}$ are the QED density and pressure corrections which can be calculated by using (14) in Appendix A. Reference [10] estimated the T_ν-T_γ relation using the electron dispersion relation method discussed in the next paragraph, but their estimate was incomplete. Equation (10) is the exact relation (assuming the neutrinos completely decouple at $T > 1$ MeV), which includes all QED corrections to the plasma and it includes the change to the coefficient $(\frac{4}{11})^{1/3}$.

How do these corrections affect BBN? QED corrections have already been applied to BBN in Ref. [10] (hereafter referred to as Dicus *et al.*), but they use a different method which does not include the total QED change to the energy density. Their approach is to calculate the finite temperature electron dispersion relation and use this to find the change in the energy density of the plasma. But one must also include the positron and *photon* dispersion relations. For example, the high T photon dispersion relation [4], $\hbar^2\omega^2 = k^2c^2 + e^2T^2/6$, is the same order of magnitude correction as the QED electron dispersion relation, and should be included. Equation (1) is very useful in that it automatically includes *all* QED corrections to the $e^{\pm}$-photon plasma without even dealing with the dispersion relations. Our result of $\Delta\rho/\rho \approx -1.1\times 10^{-3}$ is about two times greater than the result in Dicus *et al.* The correct application of the dispersion relation to the correction in pressure is discussed in more detail in Sec. V.

The primordial helium abundance Y, which is sensitive to the neutron fraction at BBN, is affected by QED in four ways: (a) the change in the electron dispersion relation (the electron mass) will affect $n \leftrightarrow p$ rates; (b) the modification of the T_ν-T_γ relation [Eq. (10)] affects $n \leftrightarrow p$ rates; (c) a decrease in the expansion rate (from $\Delta\rho$) will cause the neutron fraction to freeze-out earlier; (d) the change in the T_ν-T_γ relation will also change the time at which neutrons are (permanently) captured into deuterium, thus changing how many decay.

We can roughly estimate effects (a)–(c) by using the same analysis of Dicus *et al.*, only now including the complete correction to the equation of state. This yields $\Delta Y_a \approx 0.04 m\,\Delta m/T^2 \approx 2.9\times 10^{-4}$, $\Delta Y_b \approx = 0.15\Delta T_\nu/T_\nu \approx -0.15(7.3\times 10^{-4} - 2.9\times 10^{-4}) = -0.7\times 10^{-4}$, and $\Delta Y_c \approx 0.1\Delta\rho/\rho \approx -1.1\times 10^{-4}$. For ΔY_a, we used the value from Dicus *et al.*, since it does not need any extra correction. For ΔY_b, the first term for ΔT_ν is due to the change in the coefficient $(\frac{4}{11})^{1/3}$ in Eq. (10), and the second term is the estimate of ΔT_ν from Dicus *et al.* which does not include changes to the coefficient.

We can estimate ΔY_d by using the model in Ref. [12] to find how a small variation in the T_ν-T_γ relation affects the neutron capture time. We find that $\Delta Y_d \approx -0.03\Delta\rho/\rho \approx 0.3\times 10^{-4}$. All in all, the total effect on BBN is roughly estimated to be

$$\Delta Y_{\rm tot} = 1.4\times 10^{-4} \ . \tag{10}$$

The difference from this estimate and Dicus *et al.* ($\Delta Y_{\rm tot} \simeq 2.4\times 10^{-4}$) is directly traced to the fact that our calculation includes the complete equation of state, and that we include ΔY_d. In comparison with the observational accuracy of $\Delta Y \approx 10^{-3}$, this correction is small, but the QED corrections obtained by Dicus *et al.* are already included in the BBN codes [13], and this reanalysis shows that the correction must be changed to include the effects of the entire $e^{\pm}$-photon plasma.

One final note is that Eq. (10) also changes the present relation between the temperature of the photons and neutrinos, i.e.,

$$T_\gamma^{\rm pres} = (\tfrac{11}{4} + \delta N)^{1/3} T_\nu^{\rm pres} \tag{11}$$

because there is less energy to transfer to the photons when the $e^{\pm}$ annihilate (and the neutrinos have already decoupled). This increases the present neutrino number density $\Delta n_{\rm pres}/n = \delta N/(11/4) = 0.22\%$ per neutrino species, or on average, the present total neutrino density is about 1 neutrino/cm^3 greater.

V. THE DISPERSION RELATION

The only problem with using the diagrams in Fig. 1 to calculate the QED correction to the pressure is that these diagrams offer no immediate insight into the physical processes involved in changing the pressure. There is another way to calculate the correction by using the finite temperature dispersion relation of the electrons and photons, which does give a physically intuitive picture of these corrections.

The first correction to the electron dispersion relation is found by calculating the first correction to the electron propagator, which is represented in Fig. 5(a) [14,15]. As the caption in Fig. 5 explains, this correction is due to Compton scattering and $e^{\pm}$ annihilation and creation in the plasma. We can find the first correction to the pressure of the gas from this diagram by integrating it over all momenta p [2]. In effect, this is like joining the two free ends of the propagator and "closing the loop," making this diagram equivalent to Fig. 1(a). One must be very careful, however, to include the symmetry factors which appear in Fig. 1. The formalism of the electron propagator approach will give these factors explicitly [2,17], or one can just put them in by hand, but they must be included. One must be careful when using this method to calculate higher order terms: symmetry factors must be calculated for *each term* in the expansion. It is common to use the dispersion relation approach to find the corrections to "tree" (i.e., nonloop) interactions such as $e^- + {\rm Be} \rightarrow {\rm Li} + \nu$ described in Sec. III, but once loops are included, such as in Fig. 1, one must be careful to include the symmetry factors. There is a similar analysis for the *photon* propagator, whose first correction also comes from Compton scattering and annihilation and/or creation.

One can even take one step further in deciphering these diagrams into a physical picture. The QED correction to the electron propagator represented by Fig. 5(a), namely the inclusion of Compton scattering and $e^{\pm}$ creation and annihilation, has the effect of changing the mass of the electron. The dispersion relation takes the form [14,15] (see Appendix B)

$$E^2 = p^2c^2 + m^2c^4 + \delta m^2(p,\mu,T)c^4 \ . \tag{12}$$

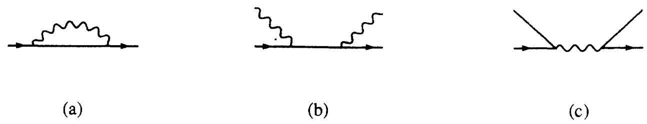

FIG. 5. The first correction to the electron propagator is 4(a). Instead of a loop, one can either cut the photon line in 4(a) and think of a photon as coming in, scattering, and then going back out into the thermal bath 4(b), or one can cut the electron line and think of a positron coming in from the plasma, annihilating with the electron, and then recreating and going back into the plasma 4(c). The sum of these two is equivalent to 4(a). Therefore, this correction is due to Compton scattering, and $e^{\pm}$ annihilation and creation. Integrating over all (thermal) momenta of the electron will close the electron line in 4(a) and form a diagram like Fig. 1(a).

Now one can use the definition of the pressure of an ideal gas,

$$P=\frac{T}{\pi^2}\int\frac{dp}{\hbar^3}p^2\ln(1\pm e^{-(E-\mu)/T})$$

for fermions and/or bosons, to relate the correction to the pressure to the change in mass:

$$\begin{aligned}P_{\rm int}^{e^2\rm term}&=\frac{1}{2}(P_E-P_{E_0})\\&=\frac{1}{2}\sum_i\frac{1}{2\pi^2}\int\frac{dp}{\hbar^3}\frac{p^2}{E_{0i}}\frac{\delta m_i^2(p,\mu,T)c^4}{e^{(E_{0i}-\mu_i)/T}\pm1}\,,\end{aligned}\tag{13}$$

where the summation is over $e^{\pm}$ and photons, and $E_{0i}^2=p^2c^2+m_i^2c^4$. One must be very careful, though, to include the coefficient of 1/2 in the right-hand side (RHS) of Eq. (13), which is the symmetry factor discussed earlier in this section. Equation (13) is equivalent to Eq. (14) in Appendix A.

To summarize this section, we have used the dispersion relation approach to gain physical insight into the QED e^2 pressure correction. The electrons gain an effective mass through Compton scattering and annihilation and/or creation, and this *change in mass is directly related to the change in pressure.* This analysis is also true for photons, whose effective mass (i.e., plasma frequency) also changes due to Compton scattering. Notice (see Fig. 6 or the Appendix) that for $T \gg mc^2$, the change in mass squared δm^2 is positive, and for $T \ll mc^2$, δm^2 is negative. This explains the change in sign of the e^2 term between Eqs. (1) and (2).

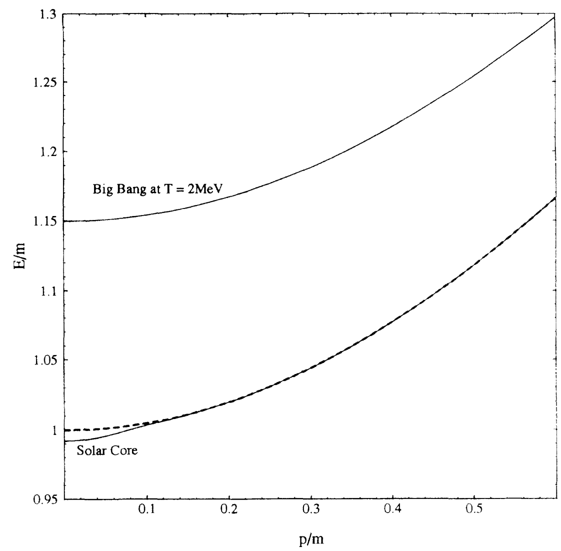

FIG 6. The electron dispersion relation in different environments. The dashed line is the vacuum dispersion relation, while the upper line is the dispersion relation for conditions in the big bang at the time of neutron decoupling ($T=2$ MeV). The lower line is for conditions in the solar core. Notice that in these two different environments, the correction to the mass of the electrons is of opposite sign.

VI. CONCLUSION

The classical calculation of the electromagnetic correction to the equation of state is not accurate in many astrophysical environments. This is not merely due to the fact that one must include relativistic or degenerate corrections, but rather there is a quantum electrodynamic correction which is important *even at nonrelativistic temperatures and nondegenerate densities,* such as in the solar core. At high temperatures, such as in the early Universe, or at high densities, it is necessary to use QED to find the largest correction to the equation of state because the classical Debye-Hückel theory breaks down.

Another perhaps more physically intuitive way to approach this problem is to use the finite temperature QED dispersion relations when calculating the pressure or energy density. By using this approach we find that Compton scattering and $e^{\pm}$ annihilation and creation are the physical processes involved in the correction. Calculating the dispersion relation has the added benefit of enabling one to calculate QED effects on reaction rates in the plasma.

This paper gives only two examples for the application of the QED correction. There are other environments, such as supernovae, which may also be significantly affected by the QED correction to the equation of state. The results for the examples of solar physics and big bang nucleosynthesis show that the QED correction is certainly large enough to warrant inclusion into solar and BBN codes.

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APPENDIX A

For convenience, we will reproduce the exact formula for the e^2 correction. This is taken directly from Ref. [4], Eq. (5.55):

$$P_{\text{int}}^{e^2\text{term}}(\mu,T)=-\frac{e^2T^2}{6}\int\frac{d^3p}{(2\pi\hbar)^3}\frac{N^-(p)+N^+(p)}{E_p}$$
$$-\frac{e^2}{2}\int\int\frac{d^3p}{(2\pi)^3}\frac{d^3q}{(2\pi)^3}\frac{e^3}{\hbar^3}$$
$$\times\frac{1}{E_p}\frac{1}{E_q}\left[\left[1+\frac{2m^2c^4}{(E_p-E_q)^2-(\mathbf{p}-\mathbf{q})^2c^2}\right][N^-(p)+N^-(q)][N^+(q)+N^+(p)]\right.$$
$$\left.+\left[1+\frac{2m^2c^4}{(E_p+E_q)^2-(\mathbf{p}-\mathbf{q})^2c^2}\right][N^-(p)+N^+(q)][N^-(q)+N^+(p)]\right], \quad \text{(A1)}$$

where $N^{\pm}(p)=1/(e^{(E\pm\mu)/T}+1)$.

APPENDIX B

The explicit form of δm^2, to order e^2, for $T \ll mc^2$ is

$$\delta m^2(p,\mu,T)=\frac{e^2T^2}{6\hbar c^5}+\frac{e^2}{m\hbar c^4}\left[\frac{mT}{2\pi}\right]^{3/2}e^{(\mu-mc^2)/T}$$
$$-\frac{e^2m}{2\pi^2\hbar e^2 p}\int_0^\infty dk k\ln\left|\frac{p+k}{p-k}\right|$$
$$\times\frac{1}{e^{(mc^2+k^2/2m-\mu)/T}+1}\,. \quad \text{(B1)}$$

If $m-\mu \gg T$ the last two terms are very small, as noted by Ref. [15], but if $m-\mu \sim T$, then at small temperatures, the third term begins to dominate and δm^2 changes sign. The change of the effective mass of the electron for finite μ was calculated by Ref. [16] but they estimated the nonrelativistic limit incorrectly, namely, they ignored the $m-\mu\sim T$ regime. The importance of the third term shows up in Fig. 5, giving the electron a negative correction to the mass.

Another important limit is $\mu \approx 0$, used in cosmology. In this case,

$$\delta m^2(p,T)=\frac{e^2T^2}{6\hbar c^5}+\frac{e^2}{\pi^2\hbar c^2}\int_0^\infty dk\frac{k^2}{E_k}\frac{1}{e^{E_k/T}+1}$$
$$-\frac{e^2m^2}{2\pi^2\hbar p}\int_0^\infty dk\frac{k}{E_k}\ln\left|\frac{p+k}{p-k}\right|\frac{1}{e^{E_k/T}+1}\,. \quad \text{(B2)}$$

The above formulas are for changes in the electron mass. The change in the "mass" of the photons is due to a plasma frequency. For $T \ll mc^2$, $\delta m_{\text{ph}}^2=e^2n_e\hbar^2/mc^4$, which is similar to the second term in (14). Since this term is very small compared to the third term in (14), we will neglect it. For $T \gg mc^2$, $\delta m_{\text{ph}}^2=e^2T^2c^3/6\hbar$ at high frequencies [4]. Note that these are all changes for *transverse* photons. The correction due to longitudinal photons (i.e., the longitudinal part of the photon self-energy [18]) is of order e^4 and so it is neglected.

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