

$$cp' = -\frac{1}{2} \hbar(\omega + \omega') + \sqrt{1 + \frac{m^2 c^4}{\hbar^2 \omega \omega'}} \frac{1}{2} \hbar(\omega - \omega')$$

$$cp' = \frac{1}{2} \hbar(\omega + \omega') + \sqrt{1 + \frac{m^2 c^4}{\hbar^2 \omega \omega'}} \frac{1}{2} \hbar(\omega - \omega')$$

$$\Rightarrow \boxed{p' - p = \frac{1}{2c} \hbar(\omega + \omega')}$$

Solving for  $\omega'$  as function of  $\omega$  at fixed  $p$ :

$$\omega' = \frac{(E - cp)\omega}{E + cp + 2\hbar\omega} \xrightarrow{\omega \rightarrow 0} 0$$

So, as the incoming light's frequency goes to zero, so does  $\omega'$  and also the difference  $p' - p \rightarrow$  in the limit  $\omega \rightarrow 0$ , we can measure  $p$  without changing its value.

(For large  $\omega$ , the measurement is not "ideal" in the sense that it changes the state of the electron; but we can still in this case determine both  $p$  before and  $p'$  after the measurement, knowing  $\omega$  and measuring  $\omega'$ . But this is only true in 1 dimensional motion.)

Let us suppose we have made a measurement of the momentum and now know that the particle is in a momentum eigenstate  $|p\rangle$ .

We can expand  $|p\rangle$  into eigenkets of the position operator  $\hat{X}$ :

$$|p\rangle = \int dx |x\rangle \langle x|p\rangle \quad (*)$$

If we make a measurement of the position and find the particle at position  $x$ , we know that after the measurement it is in the definite eigenket  $|x\rangle$ ; i.e. it is no longer in the superposition of  $\hat{X}$ -eigenkets  $(*)$ ! So even an ideal position measurement necessarily

changes the particle's state if before the  $\hat{X}$ -measurement it was in a  $\hat{P}$ -eigenstate!

Why does this happen? To measure  $x$  with perfect precision, we need to scatter light of it with infinitesimally small wavelength. According to Planck's theory of blackbody radiation and Einstein's explanation of the photo effect, such light is made of quanta that carry infinitely large energy and momentum. The recoil of the particle from the photons used to measure its position makes its momentum after the measurement completely unknown. (This is why we still calculate it from energy and momentum.)

So, while in classical mechanics an ideal measurement of any variable  $\omega$  leaves any state invariant, in quantum mechanics the <sup>ideal</sup> measurement of  $\hat{\Omega}$  leaves only the  $\hat{\Omega}$ -eigenstates invariant.

$$|\psi\rangle \xrightarrow[\omega \text{ obtained}]{\hat{\Omega} \text{ measured}} \frac{\hat{P}_\omega |\psi\rangle}{\sqrt{\langle \hat{P}_\omega \psi | \hat{P}_\omega \psi \rangle}}$$

where  $\hat{P}_\omega$  is the projection operator  $\hat{P} = \sum_{\alpha} |\omega; \alpha\rangle \langle \omega; \alpha|$  on the subspace  $\mathbb{V}_\omega$  spanned by eigenstates  $|\omega; \alpha\rangle$  of  $\hat{\Omega}$  with eigenvalue  $\omega$  (if  $\omega$  is non-degenerate, there is no sum).

Example  $\mathbb{V}^3(\mathbb{R})$ , operator  $\hat{\Omega}$  with eigenvalues  $\omega_1 = \omega_2 = \omega$ ,  $\omega_3 \neq \omega$ .

(2-fold degeneracy).

If before the measurement of  $\hat{\Omega}$ , the system is in state

$$|\psi\rangle_{(0)} = \frac{1}{2} |\omega; 1\rangle + \frac{1}{2} |\omega; 2\rangle + \frac{1}{\sqrt{2}} |\omega_3\rangle \quad (\text{normalized!})$$

and a measurement of  $\hat{\Omega}$  finds the eigenvalue  $\omega$ , then after the measurement the state vector has collapsed

to

$$|\psi\rangle_{(\omega)} \xrightarrow[\omega \text{ obtained}]{\hat{\Omega} \text{ measured}} |\psi\rangle_{(\omega)} = \frac{1}{\sqrt{2}} (|\omega; 1\rangle + |\omega; 2\rangle) \quad (\text{normalized!})$$

The component  $\sim |\omega_3\rangle$  has been "projected out" by the measurement.

If before the measurement we know nothing about the state, we have

$$|\psi\rangle_{(0)} = \alpha_1 |\omega; 1\rangle + \alpha_2 |\omega; 2\rangle + \alpha_3 |\omega_3\rangle \quad (|\alpha_1|^2 + |\alpha_2|^2 + |\alpha_3|^2 = 1)$$

After the measurement we then have

$$|\psi\rangle_{(\omega)} \xrightarrow[\omega \text{ obtained}]{\hat{\Omega} \text{ measured}} \frac{\alpha |\omega; 1\rangle + \beta |\omega; 2\rangle}{\sqrt{|\alpha|^2 + |\beta|^2}}$$

(i.e.  $|\omega_3\rangle$  was "projected out", but we don't know the probabilities for  $|\omega; 1\rangle$  and  $|\omega; 2\rangle$ ).

# Testing Quantum Theory

- To test classical mechanics, we prepare for example at time  $t=0$  a particle at position  $x_i$  with momentum  $p_i$  and then check whether at time  $t$  it arrives at the predicted position  $x_f$  with momentum  $p_f$ .
- To test quantum mechanics, we must check whether its statistical predictions are true. So we create particles in a well-defined state  $|\psi\rangle$ , then check the probabilistic predictions at later times. For this check, we have to do the experiment over and over again, with identical initial state  $|\psi\rangle$  (e.g. not with one photon, but with an entire beam of monochromatic light).

How to prepare a "well-defined initial state"  $|\psi_i\rangle$ ?

We can start with particles in an arbitrary state, then perform a set of measurements corresponding to a complete set of commuting operators (which, we know, have common eigenvectors, and a complete set of such measurements allows to specify the eigenstate completely):

$$|\psi\rangle \xrightarrow{\hat{R}_1} \frac{\hat{P}_{\omega_1}^{(1)} |\psi\rangle}{\sqrt{\langle \hat{P}_{\omega_1}^{(1)} \psi | \hat{P}_{\omega_1}^{(1)} \psi \rangle}} \xrightarrow[\substack{\hat{R}_2 \\ ([\hat{R}_1, \hat{R}_2] = 0)}]{\hat{R}_2} \frac{\hat{P}_{\omega_2}^{(2)} \hat{P}_{\omega_1}^{(1)} |\psi\rangle}{\sqrt{\langle \hat{P}_{\omega_2}^{(2)} \hat{P}_{\omega_1}^{(1)} \psi | \hat{P}_{\omega_2}^{(2)} \hat{P}_{\omega_1}^{(1)} \psi \rangle}}$$

$$\rightarrow \dots \rightarrow \underbrace{|\omega_1, \omega_2, \dots, \omega_n\rangle}_{\text{non-degenerate}}$$



After preparing the state  $|\omega_1, \omega_2, \dots, \omega_n\rangle$ , say we measure immediately (without time for the state to change) the observable  $\hat{A}$ . If, for example,

$$|\omega_1, \dots\rangle = \frac{1}{\sqrt{3}} |\lambda_1\rangle + \sqrt{\frac{2}{3}} |\lambda_2\rangle + 0 \cdot (\text{other } \lambda\text{-eigenvalues})$$

this means that the theory predicts that we will find the eigenvalue  $\lambda_1$  in  $\frac{1}{3}$  of the cases and the value  $\lambda_2$  in  $\frac{2}{3}$  of the cases, assuming we measure  $\hat{A}$  very often for particles that were all prepared in the identical state  $|\omega_1, \dots, \omega_n\rangle$ .

If we measure  $\hat{A}$  only once, we can only give a probability for finding  $\lambda_1$  or  $\lambda_2$  — the actual outcome of the measurement is otherwise unpredictable.

What we can predict with certainty is that we will never find, for particles prepared in this state, eigenvalues  $\lambda_i \neq \lambda_1, \lambda_2$ , or even  $\lambda$ -value that don't agree with any of the eigenvalues of  $\hat{A}$ !

A full corroboration of the predictions of QM thus requires a quantum ensemble of particles, all in the same state  $|\omega_1, \dots, \omega_n\rangle$ , and we must measure  $\hat{A}$  for a sufficiently large number of particles from this ensemble (where "sufficient" is defined by our requirements for how precise the test results should be).

- Of course, we can take the same quantum ensemble and measure, instead of  $\hat{A}$ , any other observable  $\hat{\Sigma}$  that we might find interesting.
- As long as the particles in the quantum ensemble don't influence each other, it doesn't matter whether we perform the  $N$  measurements of  $\hat{A}$  sequentially (i.e. by sending individual photons through a double-slit) or simultaneously (by sending a monochromatic beam of light through the same double slit).
- Is there a difference between the statistical description of, say, the outcome of throwing dice and the statistical description of the results of a quantum measurement?

Yes!

(i) In classical mechanics, it is at least in principle, possible to predict which face of the die will be up or top. In quantum mechanics the only description possible is statistical, even in principle.

(ii) If we were to write the state of the die under the mug as

$$|\psi\rangle = \sum_{i=1}^6 c_i |i\rangle \quad (|c_i|^2 = P(i))$$

where  $|i\rangle$  describes the state where the face with  $i$  dots faces up, then, when we lift the mug and find "3" pointing up, we assume in classical physics that "3" was pointing even before we lifted the mug.

In QM, the state  $|\psi_f\rangle = |3\rangle$  after the measurement is not the same as  $|\psi\rangle$  before the measurement, because our knowledge changed as a result of the measurement.

(iii) If we repeat the experiment  $N$  times, we find in the classical ensemble  $NP(1)$  dice with  $n=1$ ,  $NP(2)$  dice with  $n=2$ , etc., before and after the measurement. The quantum ensemble will contain  $N$  particles all of which are in the same state  $|\psi\rangle$  before the measurement. I.e. for each particle we have the possibility that the measurement yields any one of the 6 possible values for  $n$  pointing up. Only the ensemble before the measurement is represented by the state  $|\psi\rangle$ . After the measurement it is a mixture of 6 ensembles, each represented by a different state  $|1\rangle, |2\rangle, \dots, |6\rangle$  (with all dice in the ensemble  $|1\rangle$  showing 1 dot on their top surface).

The expectation value

$$\begin{aligned} \langle \hat{\Omega} \rangle &= \sum_i P(\omega_i) \omega_i = \sum_i |\langle \omega_i | \psi \rangle|^2 \omega_i \\ &= \sum_i \langle \psi | \omega_i \rangle \langle \omega_i | \psi \rangle \omega_i = \sum_i \langle \psi | \hat{\Omega} | \omega_i \rangle \langle \omega_i | \psi \rangle \\ &= \langle \psi | \hat{\Omega} | \psi \rangle \end{aligned}$$

So to calculate the expectation value, we only need to know the state  $|\psi\rangle$  and compute  $\langle \psi | \hat{\Omega} | \psi \rangle$ .

- In an eigenstate  $|\omega\rangle$ ,  $\hat{\Omega}$  has the expectation value  $\langle \hat{\Omega} \rangle = \omega$ .
- In an ensemble described not by an eigenstate  $|\omega\rangle$  of  $\hat{\Omega}$ ,  $\langle \hat{\Omega} \rangle$  must be obtained by a large number of measurements, each of which will yield an eigenvalue, although  $\langle \hat{\Omega} \rangle$  can be any

value (not necessarily equal to one of the eigenvalues).

The Uncertainty (standard deviation from the mean)

$$\Delta \Omega = \sqrt{\langle (\hat{\Omega} - \langle \hat{\Omega} \rangle)^2 \rangle} \quad (\text{also known as root-mean-squared (r.m.s.) deviation})$$

$$\underbrace{(\Delta \Omega)^2}_{\text{variance}} = \langle \psi | \hat{\Omega}^2 - 2\langle \hat{\Omega} \rangle \hat{\Omega} + \langle \hat{\Omega} \rangle^2 | \psi \rangle$$

$$= \sum_i \langle \psi | \hat{\Omega} | \omega_i \rangle \langle \omega_i | \hat{\Omega} | \psi \rangle$$

$$- 2\langle \hat{\Omega} \rangle \underbrace{\langle \psi | \hat{\Omega} | \psi \rangle}_{=\langle \hat{\Omega} \rangle}$$

$$+ \langle \hat{\Omega} \rangle^2 \underbrace{\langle \psi | \psi \rangle}_1 \quad (= \langle \hat{\Omega}^2 \rangle - \langle \hat{\Omega} \rangle^2)$$

$$= \sum_i \omega_i^2 \underbrace{|\langle \psi | \omega_i \rangle|^2}_{P(\omega_i)} - \underbrace{\langle \hat{\Omega} \rangle^2}_{\bar{\omega}^2} = \sum_i \underbrace{P(\omega_i)}_{\text{variance}} (\omega_i - \bar{\omega})^2$$

This is the classical statistical definition of the variance of a probability distribution  $P(\omega)$ .

Homework exercise 4.2.1, p. 129.



## Compatible and incompatible (a.k.a.

commensurate and incommensurate) variables

Considers two operators  $\hat{\Omega}$  and  $\hat{\Lambda}$ .

- If we take an ensemble of particles in some state  $|\psi\rangle$ , can we produce (by appropriate measurements) an ensemble with well-defined values  $\omega$  and  $\lambda$  for the variables described by  $\hat{\Omega}$  and  $\hat{\Lambda}$ ?
- What is the probability for obtaining such a state  $|\omega, \lambda\rangle$  when starting from the state  $|\psi\rangle$ ?

Let's first measure  $\hat{\Omega}$  in the state  $|\psi\rangle$  and select those particles where we find the eigenvalue  $\omega$ . Then we measure immediately  $\hat{\Lambda}$  and pick particles where the measurement yield the eigenvalue  $\lambda$ .

So now we have an ensemble where the particles are known to have eigenvalues  $\omega$  for  $\hat{\Omega}$  and  $\lambda$  for  $\hat{\Lambda}$ , right?

Wrong! See the example with first measuring momentum and then position. In general the second measurement alters the state obtained from the measurement and collapses the state  $|\omega\rangle = \int |\lambda\rangle \langle \lambda | \omega \rangle$  to  $|\lambda\rangle$ .

The condition that the second measurement does not change the eigenvalue  $\omega$  of  $\hat{\Omega}$  is that

$$\hat{\Omega}(\hat{\Lambda}|\omega\lambda\rangle) = \omega(\hat{\Lambda}|\omega\lambda\rangle) = \omega(\lambda|\omega\lambda\rangle)$$

i.e. that  $\hat{\Lambda}|\omega\lambda\rangle$  is an eigenstate of  $\hat{\Omega}$  with the same eigenvalue  $\omega$  as  $|\omega\lambda\rangle$  itself. So  $|\omega\lambda\rangle$  must be a simultaneous eigenvector of  $\hat{\Omega}$  and  $\hat{\Lambda}$ :

$$\hat{\Omega}|\omega\lambda\rangle = \omega|\omega\lambda\rangle$$

$$\hat{\Lambda}|\omega\lambda\rangle = \lambda|\omega\lambda\rangle$$

$$\Rightarrow \boxed{(\hat{\Omega}\hat{\Lambda} - \hat{\Lambda}\hat{\Omega})|\omega\lambda\rangle = |0\rangle}$$

So  $[\hat{\Omega}, \hat{\Lambda}]$  must have eigenkets with zero eigenvalue

3 possibilities:

A.  $[\hat{\Omega}, \hat{\Lambda}] = 0 \Rightarrow$  "Compatible": a complete basis of simultaneous eigenkets can be found.

Each such basis state describes an ensemble where  $\hat{\Omega}$  and  $\hat{\Lambda}$  have well-defined values,  $\omega$  and  $\lambda$ .

B.  $[\hat{\Omega}, \hat{\Lambda}]$  has no zero eigenvalue  $\Rightarrow$  "Incompatible"

Example:  $[\hat{X}, \hat{P}] = i\hbar$

There is no state  $|\psi\rangle \neq |0\rangle$  such that

$$[\hat{X}, \hat{P}]|\psi\rangle = i\hbar|\psi\rangle = |0\rangle$$

There is not even a single ket for which both  $\hat{X}$  and  $\hat{P}$  have well-defined values  $x$  and  $p$ .

$\rightarrow$  Heisenberg uncertainty principle:

Incompatible variables cannot be accurately determined simultaneously.

C. There are cases where  $\hat{\Omega}$  and  $\hat{\Lambda}$  don't commute but they still have some common eigenvectors, although not a complete common basis. (Why not?)

Computing the probabilities: (only interesting for case A, not C; not meaningful for case B).

If  $|\omega\rangle$  is unique (non-degenerate case)

$$P(\omega, \lambda) = |\langle \omega, \lambda | \psi \rangle|^2$$

Measuring  $\hat{\Omega}$  first:

$$|\psi\rangle = \int_{\omega} |\omega\rangle \langle \omega | \psi \rangle \quad |\omega\rangle = \int_{\lambda} |\lambda, \omega\rangle \langle \lambda, \omega | \omega \rangle$$

$$\Rightarrow |\psi\rangle = \int_{\lambda, \omega} |\lambda, \omega\rangle \underbrace{\langle \lambda, \omega | \omega \rangle}_{\hat{I}} \langle \omega | \psi \rangle = \int_{\omega} |\lambda, \omega\rangle \langle \lambda, \omega | \psi \rangle$$

$$P(\lambda, \omega) = |\langle \omega | \psi \rangle|^2 |\langle \lambda, \omega | \omega \rangle|^2 = P(\omega) P(\lambda | \omega) = |\langle \lambda, \omega | \psi \rangle|^2$$

Measuring  $\hat{\Lambda}$  first:

$$|\psi\rangle = \int_{\lambda} |\lambda\rangle \langle \lambda | \psi \rangle \quad |\lambda\rangle = \int_{\omega} |\lambda, \omega\rangle \langle \lambda, \omega | \lambda \rangle$$

$$\Rightarrow |\psi\rangle = \int_{\lambda, \omega} |\lambda, \omega\rangle \underbrace{\langle \lambda, \omega | \lambda \rangle}_{\hat{I}} \langle \lambda | \psi \rangle = \int_{\omega} |\lambda, \omega\rangle \langle \omega, \lambda | \psi \rangle$$

$$= P(\lambda, \omega) = |\langle \lambda, \omega | \lambda \rangle|^2 |\langle \lambda | \psi \rangle|^2 = P(\lambda) P(\omega | \lambda) = |\langle \lambda, \omega | \psi \rangle|^2$$

So 
$$P(\lambda, \omega) = P(\omega) P(\lambda | \omega) = P(\lambda) P(\omega | \lambda)$$
  
 conditional probabilities

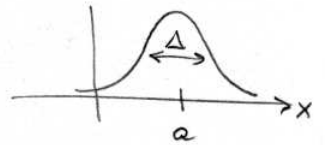
$$\Rightarrow P(\lambda | \omega) = \frac{P(\lambda)}{P(\omega)} P(\omega | \lambda)$$

$P(\lambda, \omega)$  does not depend on whether you measure  $\hat{\Omega}$  or  $\hat{\Lambda}$  first.

If the eigenvalues  $\lambda$  or  $\omega$  are degenerate, measuring  $\hat{A}$  and  $\hat{B}$  in different order will lead to different intermediate states, but the same probability  $P(\lambda, \omega)$ . If they are both degenerate, measuring both  $\lambda$  and  $\omega$  does not uniquely define the state  $|\omega, \lambda\rangle$  (there is still a multidimensional vector space spanned by all possible eigenstates  $|\omega, \lambda\rangle$ ).

Example: Gaussian wave function

Consider  $|\psi\rangle$  with  $\psi(x) \equiv \langle x|\psi\rangle = A e^{-\frac{(x-a)^2}{2\Delta^2}}$



For  $|\psi\rangle$  to be normalized, need

$$1 = \langle \psi | \psi \rangle = \int_{-\infty}^{\infty} dx \langle \psi | x \rangle \langle x | \psi \rangle = \int_{-\infty}^{\infty} dx |\psi(x)|^2 = \int_{-\infty}^{\infty} |A|^2 e^{-\frac{(x-a)^2}{\Delta^2}} dx$$

$$= |A|^2 \sqrt{\pi \Delta^2} \Rightarrow A = \frac{e^{i\alpha}}{(\pi \Delta^2)^{1/4}}$$

The overall phase factor  $e^{i\alpha}$  drops out from all probabilities, so we can set  $\alpha=0$  without loss of generality.

$$\Rightarrow \psi(x) = \frac{e^{-\frac{(x-a)^2}{2\Delta^2}}}{(\pi \Delta^2)^{1/4}}$$

Expectation value of  $\hat{X}$ :

$$\begin{aligned} \langle \hat{X} \rangle &= \langle \psi | \hat{X} | \psi \rangle = \int_{-\infty}^{\infty} dx \langle \psi | \hat{X} | x \rangle \langle x | \psi \rangle = \int_{-\infty}^{\infty} dx x |\psi(x)|^2 \\ &= \int_{-\infty}^{\infty} dy (y+a) \frac{e^{-y^2/2\Delta^2}}{\sqrt{\pi \Delta^2}} = a \int_{-\infty}^{\infty} \frac{e^{-y^2/2\Delta^2}}{\sqrt{\pi \Delta^2}} dy = a \checkmark \end{aligned}$$

↑ integrates to zero



## Uncertainty in $\hat{X}$ :

$$\begin{aligned}(\Delta X)^2 &= \langle \psi | (\hat{X} - \langle \hat{X} \rangle)^2 | \psi \rangle = \langle \psi | \hat{X}^2 - \langle \hat{X} \rangle^2 | \psi \rangle \\ &= \langle \psi | \hat{X}^2 | \psi \rangle - a^2 \langle \psi | \psi \rangle = \langle \hat{X}^2 \rangle - a^2\end{aligned}$$

$$\begin{aligned}\langle \hat{X}^2 \rangle &= \int_{-\infty}^{\infty} dx \langle \psi | \hat{X}^2 | x \rangle \langle x | \psi \rangle = \int_{-\infty}^{\infty} dx x^2 \frac{e^{-\frac{(x-a)^2}{\Delta^2}}}{\sqrt{\pi \Delta^2}} \\ &= \int_0^{\infty} dy \underbrace{(y+a)^2}_{y^2+2ay+a^2} \frac{e^{-y^2/\Delta^2}}{\sqrt{\pi \Delta^2}} = \int_0^{\infty} dy y^2 \frac{e^{-y^2/\Delta^2}}{\sqrt{\pi \Delta^2}} + 0 + a^2 \cdot 1 \\ &= \frac{\Delta^2}{2} + a^2\end{aligned}$$

$$\Rightarrow (\Delta X)^2 = \frac{\Delta^2}{2} \quad \Rightarrow \quad \boxed{\Delta X = \frac{\Delta}{\sqrt{2}}}$$

## Expectation value of $\hat{P}$ :

$$\begin{aligned}\langle \psi | \hat{P} | \psi \rangle &= \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dx' \langle \psi | x \rangle \underbrace{\langle x | \hat{P} | x' \rangle}_{-i\hbar \delta(x-x') \frac{d}{dx}} \langle x' | \psi \rangle \\ &= -i\hbar \int_{-\infty}^{\infty} dx \psi^*(x) \psi'(x) = -i\hbar \int_{-\infty}^{\infty} dx \frac{e^{-\frac{(x-a)^2}{2\Delta^2}}}{(\pi \Delta^2)^{1/4}} \left(-\frac{(x-a)}{\Delta^2}\right) \frac{e^{-\frac{(x-a)^2}{2\Delta^2}}}{(\pi \Delta^2)^{1/4}} \\ &= -i\hbar \int_{-\infty}^{\infty} dy \frac{e^{-y^2/\Delta^2}}{\sqrt{\pi \Delta^2}} \frac{y}{\Delta^2} = 0\end{aligned}$$

$$\text{Or } \langle \psi | \hat{P} | \psi \rangle = \int_{-\infty}^{\infty} dp \langle \psi | \hat{P} | p \rangle \langle p | \psi \rangle = \int_{-\infty}^{\infty} dp p |\psi(p)|^2$$

where

$$\begin{aligned}
 \psi(p) &= \langle p | \psi \rangle = \int_{-\infty}^{\infty} dx \langle p | x \rangle \langle x | \psi \rangle \\
 &= \int_{-\infty}^{\infty} dx \frac{e^{-ipx/\hbar}}{\sqrt{2\pi\hbar}} \psi(x) = \int_{-\infty}^{\infty} \frac{e^{-ipx/\hbar}}{(2\pi\hbar)^{1/2}} \frac{e^{-\frac{(x-a)^2}{2\Delta^2}}}{(\pi\Delta^2)^{1/4}} dx \\
 &= \left(\frac{\Delta^2}{\pi\hbar^2}\right)^{1/4} e^{-ipa/\hbar} e^{-p^2\Delta^2/2\hbar^2}
 \end{aligned}$$

This is a Gaussian of width  $\Delta_p = \frac{\hbar}{\Delta}$  multiplied by a phase factor  $e^{-ipa/\hbar}$  of unit magnitude.

Hence  $\int_{-\infty}^{\infty} dp p |\psi(p)|^2 = 0$  by symmetric integration

Uncertainty in  $\hat{P}$ :

Same calculation as for  $\langle X^2 \rangle$ , except for Gaussian with different width  $\Delta_p = \frac{\hbar}{\Delta}$ :

$$\Rightarrow \Delta P = \frac{\hbar}{\Delta} \frac{1}{\sqrt{2}}$$

$$\Rightarrow \Delta X \cdot \Delta P = \frac{\hbar}{2} \quad \text{a simple consequence of Fourier transforms!}$$

This happens saturate the lower bound of the uncertainty relation (see Chapter 9)

$$\Delta X \cdot \Delta P \geq \frac{\hbar}{2}$$