7.4. The harmonic oscillator in the energy basis

If we want to work in the energy basis, it appears that (in analogy to the x- and p-bases) we first must find the representation of $\hat{X}$ and $\hat{P}$ in this basis, i.e. $\langle n|\hat{X}|n'\rangle$ and $\langle n|\hat{P}|n'\rangle$, where $|n\rangle \equiv |E_n\rangle$.

We know these from the homework, but this required to first solve the $\hat{H}$-eigenvalue problem in the x-basis and then evaluating

$$\langle n|\hat{X}|n'\rangle = \int_{-\infty}^{\infty} dx\, dx' \, \langle n|x\rangle \underbrace{\langle x|\hat{X}|x'\rangle}_{x\,\delta(x-x')} \langle x'|n'\rangle$$

$$= \int_{-\infty}^{\infty} dx \, \psi_n^*(x) \times \psi_{n'}(x)$$

So the problem is already solved - why solve it again?!

Dirac discovered that you can avoid solving the EV problem in x- or p-space and solve it directly, and much more easily and elegantly, directly in the energy basis.

We start from the commutation relation

$$[\hat{X}, \hat{P}] = i\hbar \hat{I} = i\hbar$$

which is basis-independent. Now introduce (don't ask, why - you will see!)

$$\boxed{\hat{a} \equiv \frac{1}{\sqrt{2}}(\hat{\xi} + i\hat{\zeta})}$$

where $\hat{\xi} = \sqrt{\frac{m\omega}{\hbar}}\hat{X}$, $\hat{\zeta} = \frac{\hat{P}}{\sqrt{m\omega\hbar}}$; $[\hat{\xi}, \hat{\zeta}] = i$

(i.e. the unitless position and momentum operators)

$$\boxed{\hat{a}^+ = \frac{1}{\sqrt{2}}(\hat{\xi} - i\hat{\zeta})} \qquad \Rightarrow \qquad \boxed{\begin{aligned}\hat{\xi} &= \frac{1}{\sqrt{2}}(\hat{a}^+ + \hat{a}) \\ \hat{\zeta} &= \frac{i}{\sqrt{2}}(\hat{a}^+ - \hat{a})\end{aligned}}$$

They satisfy

$$\boxed{[\hat{a}, \hat{a}^+] = 1}$$

$$\Big(\text{since } [\hat{\xi}, \hat{\zeta}] = i: \ [\hat{a}, \hat{a}^+] = \frac{1}{2}[\hat{\xi} + i\hat{\zeta}, \hat{\xi} - i\hat{\zeta}] = \frac{1}{2}[i\hat{\zeta}, \hat{\xi}] - \frac{1}{2}[\hat{\xi}, i\hat{\zeta}] = +\frac{1}{2} + \frac{1}{2} = 1. \Big)$$

Now compute

$$\hat{a}^+\hat{a} = \frac{1}{2}(\hat{\xi} - i\hat{\zeta})(\hat{\xi} + i\hat{\zeta}) = \frac{1}{2}(\hat{\xi}^2 + \hat{\zeta}^2) + \frac{i}{2}[\xi, \zeta] = \frac{m\omega}{2\hbar}\hat{X}^2 + \frac{\hat{P}^2}{2m\hbar\omega} - \frac{1}{2} = \frac{\hat{H}}{\hbar\omega} - \frac{1}{2}$$

We see that

$$\boxed{\hat{H} = \left(\hat{a}^+\hat{a} + \frac{1}{2}\right)\hbar\omega}$$

What Dirac managed is to express the sum $\hat{\xi}^2 + \hat{\zeta}^2$ as a product of $\hat{\xi} + i\hat{\zeta} \sim \hat{a}$ and $\hat{\xi} - i\hat{\zeta} \sim \hat{a}^+$ $\Rightarrow$ "factorization of $\hat{H}$"

The extra $\frac{1}{2}\hbar\omega$ comes from the non-commutativity of $\hat{X} \sim \hat{\xi}$ and $\hat{P} \sim \hat{\zeta}$.

Now define the unitless Hamiltonian

$$\hat{h} \equiv \frac{\hat{H}}{\hbar\omega} = \hat{a}^+\hat{a} + \frac{1}{2}$$

Its eigenvalue problem

$$\hat{h}|\varepsilon\rangle = \varepsilon|\varepsilon\rangle$$

has as eigenvalues the energies $\varepsilon = \frac{E}{\hbar\omega}$ measured in units of $\hbar\omega$.

We observe that

$$\underset{\sim}{[\hat{a}, \hat{h}]} = [\hat{a}, \hat{a}^+\hat{a} + \tfrac{1}{2}] = [\hat{a}, \hat{a}^+\hat{a}] = \underset{\sim}{\hat{a}}$$

and

$$\underset{\sim}{[\hat{a}^+, \hat{h}]} = [\hat{a}^+, \hat{a}^+\hat{a} + \tfrac{1}{2}] = [\hat{a}^+, \hat{a}^+\hat{a}] = \hat{a}^+[\hat{a}^+, \hat{a}] = \underset{\sim}{-\hat{a}^+}$$

- The operators $\hat{a}, \hat{a}^+$ turn out to be very useful: they turn eigenstates of $\hat{H}$ into other eigenstates of $\hat{H}$!

$$\hat{h}\,\hat{a}\,|\varepsilon\rangle = (\hat{a}\hat{h} - [\hat{a}, \hat{h}])|\varepsilon\rangle = \hat{a}\,\varepsilon|\varepsilon\rangle - \hat{a}|\varepsilon\rangle$$

$$= (\varepsilon - 1)\,\hat{a}|\varepsilon\rangle$$

$\Rightarrow \hat{a}|\varepsilon\rangle$ is an eigenstate of $\hat{h}$ with eigenvalue $\varepsilon - 1$:

$$\hat{a}|\varepsilon\rangle = c_\varepsilon|\varepsilon - 1\rangle$$

(This uses the fact that in 1d there is no degeneracy of energy eigenvalues.)

- Similarly,

$$\hat{h}\hat{a}^+|\varepsilon\rangle = (\hat{a}^+\hat{h} - [\hat{a}^+, \hat{h}])|\varepsilon\rangle = \hat{a}^+\varepsilon|\varepsilon\rangle + \hat{a}^+|\varepsilon\rangle = (\varepsilon+1)\hat{a}^+|\varepsilon\rangle$$

$$\Rightarrow \hat{a}^+|\varepsilon\rangle = \tilde{c}_{\varepsilon+1}|\varepsilon+1\rangle$$

Because of these properties $\hat{a}$ and $\hat{a}^+$ are called "lowering" and "raising operators", or <u>annihilation</u> and <u>creation</u> operators because the destroy or create energy quanta with $\Delta\varepsilon = 1$ or $\Delta E = \hbar\omega$.

So if ε is an eigenvalue of $\hat{h}$, so are $\varepsilon+1, \varepsilon+2, \dots, \varepsilon+\infty$ as well as $\varepsilon-1, \varepsilon-2, \dots, \varepsilon-\infty$. But the energy can not be negative! Therefore the downward ladder must stop at some point: there must be a state $|\varepsilon_0\rangle$ which cannot be further lowered:

$$\hat{a}|\varepsilon_0\rangle = 0$$

$$\Rightarrow \hat{a}^+\hat{a}|\varepsilon_0\rangle = 0 \quad \Rightarrow \hat{h}|\varepsilon_0\rangle = \tfrac{1}{2}|\varepsilon_0\rangle \quad \Rightarrow \boxed{\varepsilon_0 = \tfrac{1}{2}}$$

Starting from this ground state, repeated application of $\hat{a}^+$ will raise the energy indefinitely, giving eigenvalues

$$\boxed{\varepsilon_n = n + \tfrac{1}{2} \quad \Rightarrow E_n = (n+\tfrac{1}{2})\hbar\omega \quad (n=0,1,2,\dots)}$$

Are there any other eigenstates? Let us assume that another ladder of states, with a different ground state $|\varepsilon_0'\rangle$, exists: $\hat{a}|\varepsilon_0'\rangle = 0$

$$\Rightarrow \hat{a}^+\hat{a}|\varepsilon_0'\rangle = 0 \Rightarrow \hat{h}|\varepsilon_0'\rangle = \tfrac{1}{2}|\varepsilon_0'\rangle$$

So $|\varepsilon_0'\rangle$ has the same energy eigenvalue as $|\varepsilon_0\rangle$.

But in 1d there are no degenerate bound states:

Let ψ_1, ψ_2 be two eigenfunctions of $\hat{H}$ with the same eigenvalue E:

$$\left.\begin{aligned} -\frac{\hbar^2}{2m}\psi_1'' + V\psi_1 &= E\psi_1 \\ -\frac{\hbar^2}{2m}\psi_2'' + V\psi_2 &= E\psi_2 \end{aligned}\right\} \psi_1\psi_2'' - \psi_2\psi_1'' = 0$$

$$\Rightarrow \frac{d}{dx}\left(\psi_1\psi_2' - \psi_2\psi_1'\right) = 0 \quad \text{or} \quad \psi_1 \frac{d\psi_2}{dx} = \psi_2 \frac{d\psi_1}{dx}$$

$$\Rightarrow \frac{d\psi_1}{\psi_1} = \frac{d\psi_2}{\psi_2} \quad \text{or} \quad \log\psi_1 = \log\psi_2 + C$$

$$\Rightarrow \psi_1 = e^C \psi_2$$

The two solutions can only differ by a scale factor and thus represent the same state $\rightarrow$ no degeneracy.

$\Rightarrow$ $|\varepsilon_0\rangle$ and $|\varepsilon_0'\rangle$ represent the same state; once normalized, they can differ only by an irrelevant phase.

Let us now see how we can compute the constants C_ε and $\tilde{C}_{\varepsilon+1}$:

Since $\varepsilon = n + \frac{1}{2}$, let us label the states by n:

$|\varepsilon_0\rangle \rightarrow |0\rangle$ (this is not the null vector!)

$|\varepsilon_n\rangle \rightarrow |n\rangle$

and write $\hat{a}|n\rangle = C_n|n-1\rangle$, $\hat{a}^+|n\rangle = \tilde{C}_{n+1}|n+1\rangle$

Consider the adjoint:

$$\langle n|\hat{a}^+ = \langle n-1|C_n^*$$

and $$\underbrace{\langle n|\hat{a}^+\hat{a}|n\rangle}_{\hat{h}-\frac{1}{2}} = \underbrace{\langle n-1|n-1\rangle}_{1} C_n^* C_n = |C_n|^2$$

Using $\hat{h}|n\rangle = \varepsilon_n|n\rangle = (n+\frac{1}{2})|n\rangle$

$$\Rightarrow |C_n|^2 = \langle n|n|n\rangle = n\langle n|n\rangle = n$$

$$\Rightarrow C_n = \sqrt{n}\, e^{i\phi} \quad (\phi \text{ arbitrary})$$

We can choose $\phi = 0$.

$$\Rightarrow \boxed{\hat{a}|n\rangle = \sqrt{n}\,|n-1\rangle}$$

Similarly:

$$\hat{a}^+|n\rangle = \tilde{c}_{n+1}|n+1\rangle$$

$$\langle n|\underbrace{\hat{a}\hat{a}^+}_{=1+\hat{a}^+\hat{a}}|n\rangle = \langle n+1|n+1\rangle|\tilde{c}_{n+1}|^2$$

$$\Rightarrow |\tilde{c}_{n+1}|^2 = 1 + \langle n|\hat{a}^+\hat{a}|n\rangle = 1+n$$

$$\Rightarrow \boxed{\hat{a}^+|n\rangle = \sqrt{n+1}\,|n+1\rangle}$$

Finally

$$\hat{a}^+\hat{a}|n\rangle = \hat{a}^+\sqrt{n}\,|n-1\rangle = \sqrt{n}\left(\sqrt{n}\,|n\rangle\right) = n|n\rangle$$

So $|n\rangle$ is as eigenstate of the "number operator"

$$\hat{N} \equiv \hat{a}^+\hat{a}$$

Which counts the number of energy quanta $\hbar\omega$ in state $|n\rangle$ in excess of the ground state energy $\frac{1}{2}\hbar\omega$.

We see that

$$\boxed{\hat{h} = \hat{N} + \frac{1}{2} \quad \text{and} \quad \hat{H} = \hbar\omega\left(\hat{N}+\frac{1}{2}\right)}$$

We can now compute in 5 seconds

$$\langle n'|\hat{a}|n\rangle = \langle n'|n-1\rangle\sqrt{n} = \sqrt{n}\,\delta_{n',n-1}$$

$$\langle n'|\hat{a}^+|n\rangle = \langle n'|n+1\rangle\sqrt{n+1} = \sqrt{n+1}\,\delta_{n',n+1}$$

from which we get (remember

$$\hat{X} = \sqrt{\frac{\hbar}{m\omega}}\,\hat{\xi} = \sqrt{\frac{\hbar}{2m\omega}}\,(\hat{a}^+ + \hat{a}), \quad \hat{P} = \sqrt{m\hbar\omega}\,\hat{\zeta} = i\sqrt{\frac{m\hbar\omega}{2}}\,(\hat{a}^+ - \hat{a})\Big)$$

$$\boxed{\begin{aligned} \langle n'|\hat{X}|n\rangle &= \sqrt{\frac{\hbar}{2m\omega}}\left(\sqrt{n+1}\,\delta_{n',n+1} + \sqrt{n}\,\delta_{n',n-1}\right) \\ \langle n'|\hat{P}|n\rangle &= i\sqrt{\frac{m\hbar\omega}{2}}\left(\sqrt{n+1}\,\delta_{n',n+1} - \sqrt{n}\,\delta_{n',n-1}\right) \end{aligned}}$$

(In the x-basis, this calculation (Exercise 7.3.4, p. 196) required >15 minutes!)

In the energy basis $\{|n\rangle, n=0,1,2,\dots\}$ we can thus represent $\hat{a}$ and $\hat{a}^+$ by the matrices

$$\hat{a}^+ \longleftrightarrow \begin{array}{c|c} & n=0 \quad n=1 \quad n=2 \quad \dots \\ \begin{array}{c} n'=0 \\ n'=1 \\ n'=2 \\ \vdots \end{array} & \begin{pmatrix} 0 & & & & \\ \sqrt{1} & 0 & & 0 & \\ & \sqrt{2} & 0 & & \\ & & \sqrt{3} & 0 & \\ & 0 & & & \ddots \end{pmatrix} \end{array}$$

$$\hat{a} \leftrightarrow \begin{array}{c|c} & n=0 \quad n=1 \quad n=3 \quad n=4 \\ \begin{array}{c} n'=0 \\ n'=1 \\ n'=2 \\ n'=4 \\ \vdots \end{array} & \begin{pmatrix} 0 & \sqrt{1} & & & \\ & 0 & \sqrt{2} & & 0 \\ & & 0 & \sqrt{3} & \\ & & & 0 & \sqrt{4} \\ & 0 & & & \ddots \end{pmatrix} \end{array}$$

and $\hat{X}$ and $\hat{P}$ by

$$\hat{X} \leftrightarrow \sqrt{\frac{\hbar}{2m\omega}} \begin{pmatrix} 0 & \sqrt{1} & & & & \\ -\sqrt{1} & 0 & \sqrt{2} & & 0 & \\ & \sqrt{2} & 0 & \sqrt{3} & & \\ & & \sqrt{3} & 0 & \sqrt{4} & \\ & & & -\sqrt{4} & 0 & \\ & 0 & & & & \ddots \end{pmatrix}$$

$$\hat{P} \leftrightarrow i\sqrt{\frac{m\hbar\omega}{2}} \begin{pmatrix} 0 & -\sqrt{1} & & & & \\ \sqrt{1} & 0 & -\sqrt{2} & & 0 & \\ & \sqrt{2} & 0 & -\sqrt{3} & & \\ & & -\sqrt{3} & 0 & -\sqrt{4} & \\ & & & \sqrt{4} & 0 & \\ & 0 & & & & 0 \end{pmatrix}$$

while the Hamiltonian is

$$\hat{H} \leftrightarrow \hbar\omega \begin{pmatrix} 1/2 & & & & 0 \\ & 3/2 & & & \\ & & 5/2 & & \\ & & & 7/2 & \\ 0 & & & & \ddots \end{pmatrix}$$

The state $|n\rangle$ can be expressed directly through the ground state by applying $\hat{a}^+$ n-times:

$$|n\rangle = \frac{\hat{a}^+}{\sqrt{n}}|n-1\rangle = \frac{\hat{a}^+}{\sqrt{n}}\frac{\hat{a}^+}{\sqrt{n-1}}|n-2\rangle = \dots = \frac{(\hat{a}^+)^n}{\sqrt{n!}}|0\rangle$$

To see the power of the energy basis, consider computing $\langle 3|\hat{P}^3|2\rangle$ in the x-basis:

$$\langle 3|\hat{P}^3|2\rangle = \sqrt{\frac{m\omega}{\pi\hbar}}\sqrt{\frac{1}{2^3\cdot 3!}\,\frac{1}{2^2\cdot 2!}}\int_{-\infty}^{\infty} dx\, e^{-\frac{m\omega x^2}{2\hbar}} H_3\left(\sqrt{\frac{m\omega}{\hbar}}x\right)(-i\hbar)^3 \cdot \frac{d^3}{dx^3}\left(H_2\left(\sqrt{\frac{m\omega}{\hbar}}x\right)e^{-\frac{m\omega x^2}{2\hbar}}\right)$$

and then compare it with the energy basis:

$$\langle 3|\hat{P}^3|2\rangle = i^3\left(\frac{m\hbar\omega}{2}\right)^{3/2}\langle 3|(\hat{a}^+ - \hat{a})^3|2\rangle$$

$$= -i\left(\frac{m\hbar\omega}{2}\right)^{3/2}\langle 3|(\hat{a}^+)^3 - (\hat{a}^+)^2\hat{a} - \hat{a}^+\hat{a}\hat{a}^+ - \hat{a}(\hat{a}^+)^2$$
$$+ \hat{a}^+\hat{a}^2 + \hat{a}\hat{a}^+\hat{a} + \hat{a}^2\hat{a}^+ - \hat{a}^3|2\rangle$$

(We have to watch out to keep the $\hat{a}, \hat{a}^+$ operators in the correct order!)

Since $\hat{a}$ lowers n by 1 and $\hat{a}^+$ raises n by 1, and we want to go up by 1 from 2 to 3 in order to get a non-zero matrix element, we need only the terms containing one $\hat{a}$ and two $\hat{a}^+$:

$$(\hat{a}^+)^2\hat{a}|2\rangle = \sqrt{2}(\hat{a}^+)^2|1\rangle = \sqrt{2}\,\hat{a}^+\sqrt{2}|2\rangle = 2\cdot\sqrt{3}\,|3\rangle$$

$$\hat{a}^+\hat{a}\hat{a}^+|2\rangle = \hat{a}^+\hat{a}\sqrt{3}\,|3\rangle = \sqrt{3}\,\hat{a}^+\sqrt{3}\,|2\rangle = 3\sqrt{3}\,|3\rangle$$

$$\hat{a}(\hat{a}^+)^2|2\rangle = \hat{a}\hat{a}^+\sqrt{3}\,|3\rangle = \sqrt{3}\,\hat{a}\sqrt{4}\,|4\rangle = \sqrt{3\cdot 4}\sqrt{4}\,|3\rangle = 4\sqrt{3}\,|3\rangle$$

$$\Rightarrow \langle 3|\hat{P}^3|2\rangle = -i\left(\frac{m\hbar\omega}{2}\right)^{3/2}\Big(\underbrace{-2\sqrt{3} - 3\sqrt{3} - 4\sqrt{3}}_{-9\sqrt{3} = -3^{3/2}\cdot 3}\Big) =$$

$$= 3i\left(\frac{3}{2}m\hbar\omega\right)^{3/2}$$

Much faster and much simpler algebra!

Let us note a remarkable feature of Dirac's solution of the harmonic oscillator problem: it is completely algebraic — no solution of differential equations!

The key to the solution was the merciless exploitation of the commutator $[\hat{X}, \hat{P}] = i\hbar$. Dirac noted the close connection of the commutator in quantum mechanics with the Poisson bracket in classical mechanics:

for $\omega(x_i, p_j)$, $\lambda(x_i, p_j)$,

$$\{\omega, \lambda\} \equiv \sum_i \left(\frac{\partial \omega}{\partial x_i} \frac{\partial \lambda}{\partial p_i} - \frac{\partial \omega}{\partial p_i} \frac{\partial \lambda}{\partial x_i} \right)$$

Specifically, $\{x, p\} = \frac{\partial x}{\partial x} \frac{\partial p}{\partial p} - \frac{\partial x}{\partial p} \frac{\partial p}{\partial x} = 1$

So we have

$$\boxed{[\hat{X}, \hat{P}] = i\hbar \{x, p\} = i\hbar}$$

This leads to an immediate generalization to N degrees of freedom:

<u>Postulate II</u>. The classical Cartesian coordinates $x_1, \dots, x_N$ and momenta $p_1, \dots, p_N$ of a system with N degrees of freedom are replaced in Quantum Mechanics by Hermitean operators $\hat{X}_1, \dots, \hat{X}_N$ and $\hat{P}_1, \dots, \hat{P}_N$ obeying commutation relations

$$[\hat{X}_k, \hat{P}_\ell] = i\hbar \{x_k, p_\ell\} = i\hbar \delta_{k\ell}$$

$$[\hat{X}_k, \hat{X}_\ell] = i\hbar \{x_k, x_\ell\} = 0$$

$$[\hat{P}_k, \hat{P}_\ell] = i\hbar \{p_k, p_\ell\} = 0$$

It turns out that if

$$\{\omega(x,p), \lambda(x,p)\} = \gamma(x,p)$$ (Poisson brackets of classical observables)

then

$$\left[\hat{\Omega}(\hat{X},\hat{P}), \Lambda(\hat{X},\hat{P})\right] = i\hbar\, \hat{\Gamma}(\hat{X},\hat{P})$$

(modulo ordering ambiguities which occur between non-commuting operators but not in the classical case).

7.5 Passage from energy basis to $\hat{X}$-basis

To compute the $\overbrace{\text{probability to find}}^{\rho_n(x)=|\langle x|n\rangle|^2}$ the particle at position x, we need $\psi_n(x) = \langle x|n\rangle$.

This can be done easily as follows:

Let's begin with the ground state $|0\rangle$, satisfying $\hat{a}|0\rangle = 0$:

$$|0\rangle \longrightarrow \langle x|0\rangle = \psi_0(x)$$

$$\hat{a} = \sqrt{\frac{m\omega}{2\hbar}}\,\hat{X} + i\sqrt{\frac{1}{2m\hbar\omega}}\,\hat{P} \longrightarrow \sqrt{\frac{m\omega}{2\hbar}}\,x + \sqrt{\frac{\hbar}{2m\omega}}\,\frac{d}{dx} = \frac{1}{\sqrt{2}}\left(\xi + \frac{d}{d\xi}\right)$$

$$\hat{a}^+ \longrightarrow \frac{1}{\sqrt{2}}\left(\xi - \frac{d}{d\xi}\right)$$

$$\hat{a}|0\rangle = 0 \longrightarrow \left(\xi + \frac{d}{d\xi}\right)\psi_0(\xi) = 0$$

$$\Rightarrow -\xi\, d\xi = \frac{d\psi_0}{\psi_0} = d\ln\psi_0$$

$$\Rightarrow \boxed{\psi_0(\xi) = A_0 e^{-\xi^2/2}}$$

$$\text{or } \psi_0(x) = A_0\, e^{-\frac{m\omega x^2}{2\hbar}}$$

After normalization $\longrightarrow$ $\psi_0(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-\frac{m\omega x^2}{2\hbar}}$

Next

$$|n\rangle = \frac{(\hat{a}^+)^n}{\sqrt{n!}}|0\rangle \longrightarrow \langle x|n\rangle = \psi_n(x) = \frac{1}{\sqrt{n!}}\left(\frac{\xi - \frac{d}{d\xi}}{\sqrt{2}}\right)^n \langle x(\xi)|0\rangle$$

$$= \frac{1}{\sqrt{n!}}\frac{1}{\sqrt{2^n}}\left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\left(\xi - \frac{d}{d\xi}\right)^n e^{-\xi^2/2}$$

Comparing this with the already computed coordinate-space eigenfunction we find a useful identity:

$$\boxed{H_n(\xi) = e^{\xi^2/2}\left(\xi - \frac{d}{d\xi}\right)^n e^{-\xi^2/2}} \qquad (*)$$

$$\left(\text{E.g. } H_1(\xi) = e^{\xi^2/2}\left(\xi - \frac{d}{d\xi}\right)e^{-\xi^2/2} = 2\xi \checkmark\right)$$

Putting it differently: if we take (*) from a textbook on special functions as the definition of the Hermite polynomials, the above expression for $\psi_n(x)$ gives us the desired normalization constant

$$A_n = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\sqrt{\frac{1}{2^n \cdot n!}} = \left(\frac{m\omega}{\pi\hbar\, 2^{2n}(n!)^2}\right)^{1/4}$$

Computing it from $|n\rangle = \frac{(\hat{a}^+)^n}{\sqrt{n!}}|0\rangle$ is much easier than normalizing the integral $\int_{-\infty}^{\infty}|\psi_n(x)|^2 dx$.