7.6. Classical ("coherent") states in the harmonic oscillator

We already saw the "correspondence principle" in action: Classical behavior appears to re-emerge from quantum mechanics in the limit of very large quantum numbers (e.g. $n \to \infty$ in the harmonic oscillator) where the discreteness of the energy spectrum and the finite level spacing between energy levels can be ignored. However, a much tighter connection with classical dynamics can be made through "coherent states" which are minimum uncertainty wavepackets sloshing back and forth between the potential walls, without spreading. We will illustrate this for the harmonic oscillator.

We first observe that the harmonic oscillator eigenstates satisfy $\boxed{(\Delta P \cdot \Delta X)_{|n\rangle} = (n+\tfrac{1}{2})\hbar}$ $(n = 0, 1, 2, \ldots)$ (Exercise 7.4.2 (p.212))

So the higher-lying excited states are certainly not minimum uncertainty states – in fact, as $n \to \infty$, the uncertainty product of the energy eigenstates approaches infinity!

Thus, if we want to describe, say, a classical pendulum at small amplitude (where the harmonic oscillator approximation works well) in terms of quantum mechanical states, harmonic oscillator eigenstates seem a poor choice.

What to choose instead? Well, we learned that Gaussian wave packets in general have a minimal uncertainty product. So let us make an ansatz

$$\psi(x,t=0) = C\, e^{ip_0 x/\hbar}\, e^{-\frac{m\omega(x-x_0)^2}{2\hbar}}$$

where $p_0 = \langle \hat{P} \rangle_{t=0}$ and $x_0 = \langle \hat{X} \rangle_{t=0}$ are the mean momentum and position of the pendulum at $t=0$.
The width in x has been adjusted to correspond to that of the harmonic oscillator ground state.

Let us work out $(m\omega \hat{X} + i\hat{P})|\psi(t=0)\rangle$ in the x-basis:

$$\left(m\omega x + \hbar \frac{d}{dx}\right) C e^{ip_0 x/\hbar} e^{-m\omega(x-x_0)^2/2\hbar} =$$

$$= \left(m\omega x + \hbar\left(ip_0/\hbar - m\omega(x-x_0)/\hbar\right)\right) C e^{ip_0 x/\hbar} e^{-m\omega(x-x_0)^2/2\hbar}$$

$$= (m\omega x_0 + ip_0)\, C e^{ip_0 x/\hbar} e^{-m\omega(x-x_0)^2/2\hbar}$$

$$\Rightarrow \boxed{\left(m\omega \hat{X} + i\hat{P}\right)|\psi(t=0)\rangle = (m\omega x_0 + ip_0)|\psi(t=0)\rangle}$$

This is an eigenvalue equation: $|\psi(t=0)\rangle$ is an eigenstate of $m\omega\hat{X} + i\hat{P}$ with eigenvalue $m\omega x_0 + ip_0$!

- Now,

$$m\omega\hat{X} + i\hat{P} = \sqrt{m\omega\hbar}\left(\sqrt{\frac{m\omega}{\hbar}}\hat{X} + i\frac{\hat{P}}{\sqrt{m\hbar\omega}}\right) = \sqrt{m\omega\hbar}\left(\hat{\xi} + i\hat{\zeta}\right)$$

$$= \sqrt{2m\omega\hbar}\,\hat{a}$$

$\Rightarrow |\psi(t=0)\rangle$ is an eigenstate of $\hat{a}$, with eigenvalue

$$z_0 \equiv \frac{1}{\sqrt{2}}(\xi_0 + i\zeta_0) \equiv \frac{1}{\sqrt{2}}\left(\sqrt{\frac{m\omega}{\hbar}}\, x_0 + \frac{i p_0}{\sqrt{m\hbar\omega}}\right):$$

$$\boxed{\hat{a}\,|\psi(t=0)\rangle \equiv \hat{a}\,|z_0\rangle = z_0\,|z_0\rangle}$$

- Now let's look at how this state evolves:

$$|\psi(t)\rangle = e^{-i\hat{H}t/\hbar}\,|\psi(t=0)\rangle = e^{-i(\hat{a}^+\hat{a}+\frac{1}{2})\omega t}\,|\psi(t=0)\rangle$$

Let us compute the action of $\hat{a}$ on this evolved state:

$$\hat{a}\,|\psi(t)\rangle = \hat{a}\,e^{-i\hat{H}t/\hbar}|z_0\rangle = e^{-i\hat{H}t/\hbar}\underbrace{\left(e^{i\hat{H}t/\hbar}\,\hat{a}\,e^{-i\hat{H}t/\hbar}\right)}_{\hat{a}(t)}|z_0\rangle$$

We can work out $\hat{a}(t)$: Since $\hat{H}|n\rangle = (n+\frac{1}{2})\hbar\omega|n\rangle$, we have

$$\langle m|\hat{a}(t)|n\rangle = e^{i(m+\frac{1}{2})\omega t}\langle m|\hat{a}|n\rangle\, e^{-i(n+\frac{1}{2})\omega t}$$

$$= e^{i(m-n)\omega t}\langle n-1|\hat{a}|n\rangle\,\delta_{m,n-1} = e^{-i\omega t}\langle n-1|\hat{a}|n\rangle\delta_{m,n-1}$$

$$= e^{-i\omega t}\langle m|a|n\rangle$$

This is true for all m, n $\rightarrow$

$$\boxed{\hat{a}(t) = e^{i\hat{H}t/\hbar}\,\hat{a}\,e^{-i\hat{H}t/\hbar} = e^{-i\omega t}\,\hat{a}}$$

Therefore

$$\hat{a}\,|\psi(t)\rangle = e^{-i\hat{H}t/\hbar}\, e^{-i\omega t}\,\hat{a}\,|\psi(t=0)\rangle$$

$$= e^{-i\omega t}\, e^{-i\hat{H}t/\hbar}\, z_0\,|\psi(t=0)\rangle$$

$$= \left(z_0 e^{-i\omega t}\right)|\psi(t)\rangle$$

Hence the wave function at later times obeys the same eigenvalue equation as at time 0, only with a different eigenvalue:

$$\boxed{z(t) = e^{-i\omega t} z_0 = e^{-i\omega t} \frac{\xi_0 + i\zeta_0}{\sqrt{2}}}$$

Separating this into real and imaginary parts and translating back from ξ, ζ to x, p we get

$$\boxed{\begin{aligned} x(t) &= x_0 \cos(\omega t) + \frac{p_0}{m\omega} \sin(\omega t) = \langle \hat{X} \rangle(t) \\ p(t) &= p_0 \cos(\omega t) - m\omega x_0 \sin(\omega t) = \langle \hat{P} \rangle(t) \end{aligned}}$$

i.e. just the classical trajectory of the pendulum bob!

Furthermore, we can show that the coherent state $|\psi(t=0)\rangle$ remains a minimum-uncertainty state at later times: It's wave function satisfies the eigenvalue equation

$$\left(m\omega x + \hbar \frac{d}{dx}\right)\psi(x,t) = \left(m\omega \langle\hat{X}\rangle(t) + i\langle\hat{P}\rangle(t)\right)\psi(x,t)$$

which is solved by

$$\boxed{\psi(x,t) = C e^{i\langle\hat{P}\rangle(t)\, x/\hbar}\, e^{-\frac{m\omega}{2\hbar}\left(x - \langle\hat{X}\rangle(t)\right)^2}}$$

i.e. a Gaussian wave packet with the <u>same</u> width as at $t=0$, but with moving center $\langle\hat{X}\rangle(t)$ and time dependent velocity of its peak position, $v(t) = \frac{\langle\hat{P}\rangle(t)}{m}$.

This was discovered by Schrödinger; it is the quantum state with the most similarity to a classical particle in the harmonic oscillator.

These coherent states have some intriguing mathematical properties. Most importantly, I am now going to show that

$$\boxed{|z_0\rangle = e^{z_0\hat{a}^+ - z_0^*\hat{a}}|0\rangle}$$

where $|0\rangle$ is the harmonic oscillator ground state, $\hat{a}|0\rangle = 0$.

Using $e^{z_0\hat{a}^+ - z_0^*\hat{a}} = e^{z_0\hat{a}^+} e^{-z_0^*\hat{a}} e^{-\frac{1}{2}[z_0\hat{a}^+, -z_0^*\hat{a}]}$

(since $[\hat{a}^+, \hat{a}] = -1$ commutes with everything)

$$= e^{-|z_0|^2/2} e^{z_0\hat{a}^+} e^{-z_0^*\hat{a}}$$

we can rewrite this as

$$\boxed{|z_0\rangle = e^{-|z_0|^2/2} e^{z_0\hat{a}^+}\Big(\underbrace{e^{-z_0^*\hat{a}}|0\rangle}_{=|0\rangle}\Big) = e^{-|z_0|^2/2} e^{z_0\hat{a}^+}|0\rangle}$$

This gives a rule for explicit construction of $|z_0\rangle$ from the harmonic oscillator ground state. It is clear that a coherent state is a superposition of states with any number of phonons $\hbar\omega$:

$$\underset{\sim}{|z_0\rangle} = e^{-|z_0|^2/2}\sum_{n=0}^{\infty}\frac{(z_0 a^+)^n}{n!}|0\rangle = e^{-|z_0|^2/2}\sum_{n=0}^{\infty}\frac{z_0^n}{\sqrt{n!}}\underbrace{\left(\frac{(\hat{a}^+)^n}{\sqrt{n!}}|0\rangle\right)}_{|n\rangle}$$

$$= \underbrace{e^{-|z_0|^2/2}\left(|0\rangle + z_0|1\rangle + \frac{z_0^2}{\sqrt{2!}}|2\rangle + \frac{z_0^3}{\sqrt{3!}}|3\rangle + \ldots\right)} = e^{-|z_0|^2/2}\sum_{n=0}^{\infty}\frac{z_0^n}{\sqrt{n!}}|n\rangle$$

Let us now check that $|z_0\rangle$ satisfies the eigenvalue equation $\hat{a}|z_0\rangle = z_0|z_0\rangle$:

$$\hat{a}|z_0\rangle = e^{-|z_0|^2/2}\,\hat{a}\,e^{z_0\hat{a}^+}|0\rangle = e^{-|z_0|^2/2}\left[\hat{a}, e^{z_0\hat{a}^+}\right]|0\rangle$$

(since $\hat{a}|0\rangle = 0$)

Now $\underbrace{\left[\hat{a}, e^{z_0\hat{a}^+}\right]} = \hat{a} e^{z_0\hat{a}^+} - e^{z_0 a^+}\hat{a}$

$$= e^{z_0\hat{a}^+}\left(\underbrace{e^{-z_0\hat{a}^+}\hat{a}\, e^{z_0\hat{a}^+}} - \hat{a}\right)$$

$$= e^{z_0\hat{a}^+}\left(\hat{a} + z_0\underbrace{[\hat{a},\hat{a}^+]}_{1} - \hat{a}\right) = z_0 e^{z_0\hat{a}^+}$$

$\rightarrow e^{-\lambda\hat{A}}\hat{B}e^{\lambda\hat{A}} = \hat{B} - \lambda[\hat{A},\hat{B}] + \frac{\lambda^2}{2!}[\hat{A},[\hat{A},\hat{B}]] - \frac{\lambda^3}{3!}[\hat{A},[\hat{A},[\hat{A},\hat{B}]]] + \dots$

$$\Rightarrow \quad \hat{a}|z_0\rangle = z_0 e^{-|z_0|^2/2} e^{z_0\hat{a}^+}|0\rangle = z_0|z_0\rangle \checkmark$$

So, indeed, $|z_0\rangle = e^{-|z_0|^2/2} e^{z_0 a^+}|0\rangle = e^{z_0\hat{a}^+ - z_0^*\hat{a}}|0\rangle$

is our coherent state, and we have

$$|z(t)\rangle = \lambda(t)|z_0 e^{-i\omega t}\rangle = \lambda e^{z_0 e^{-i\omega t}\hat{a}^+ - z_0^* e^{i\omega t}\hat{a}}|0\rangle$$

$$= \lambda e^{-|z_0|^2/2} e^{z_0 e^{-i\omega t}\hat{a}^+}|0\rangle$$

for phase factor $\lambda(t)$ see 168' !

The mean position and momentum in the coherent state are

$$\langle z|\hat{X}|z\rangle = \sqrt{\frac{\hbar}{2m\omega}}\langle z|\hat{a}^+ + \hat{a}|z\rangle = \sqrt{\frac{\hbar}{2m\omega}}(z^* + z) = \sqrt{\frac{2\hbar}{m\omega}}\,\mathrm{Re}\, z$$

$$= \sqrt{\frac{\hbar}{m\omega}}\,\xi = x \quad \Rightarrow \quad \langle z(t)|\hat{X}|z(t)\rangle = x(t) = x_0\cos\omega t + \frac{p_0}{m\omega}\sin\omega t$$

$$\langle z|\hat{P}|z\rangle = i\sqrt{\frac{\hbar m\omega}{2}}\langle z|a^+ - \hat{a}|z\rangle = i\sqrt{\frac{\hbar m\omega}{2}}(z^* - z) = \sqrt{2m\hbar\omega}\,\mathrm{Im}\, z$$

$$= \sqrt{m\hbar\omega}\,\zeta = p \quad \Rightarrow \quad \langle z(t)|\hat{P}|z(t)\rangle = p(t) = p_0\cos\omega t - m\omega x_0\sin\omega t$$

Furthermore

$$\langle z|\hat{X}^2|z\rangle = \frac{\hbar}{2m\omega}\langle z|\,\hat{a}^2 + (\hat{a}^+)^2 + \hat{a}^+\hat{a} + \underbrace{\hat{a}\hat{a}^+}_{=1+\hat{a}^+\hat{a}}|z\rangle$$

$$= \frac{\hbar}{2m\omega}\langle z|\,z^2 + (z^*)^2 + 2z^*z + 1|z\rangle = \frac{\hbar}{2m\omega}(z+z^*)^2 + \frac{\hbar}{2m\omega}$$

$$= \langle z|\hat{X}|z\rangle^2 + \frac{\hbar}{2m\omega} \quad \Rightarrow (\Delta X)^2 = \langle z|\hat{X}^2|z\rangle - \langle z|\hat{X}|z\rangle^2 = \frac{\hbar}{2m\omega}$$

$$\langle z|\hat{P}^2|z\rangle = \dots = \langle z|\hat{P}|z\rangle^2 + \frac{\hbar m\omega}{2} \quad \Rightarrow (\Delta P)^2 = \langle z|\hat{P}^2|z\rangle - \langle z|\hat{P}|z\rangle^2 = \frac{\hbar m\omega}{2}$$

$$\Rightarrow \boxed{\Delta P\cdot\Delta X = \frac{\hbar}{2} \quad \forall t} \quad \text{(as already seen)}$$

$$|z(t)\rangle = \hat{U}(t)|z_0\rangle = \hat{U}(t)\left(e^{-|z_0|^2/2}\, e^{z_0\hat{a}^+}|0\rangle\right)$$

$$= e^{-|z_0|^2/2}\, \hat{U}(t)\, e^{z_0\hat{a}^+}\, \hat{U}^+(t)\, \underbrace{\left(\hat{U}(t)|0\rangle\right)}_{\substack{= e^{-i/\hbar \hat{H} t}|0\rangle = e^{-i/\hbar(\frac{1}{2}\hbar\omega)t}|0\rangle \\ = e^{-i\omega t/2}|0\rangle}}$$

$$= e^{-i\omega t/2}\, e^{-|z_0|^2/2}\, e^{z_0 \hat{U}(t)\hat{a}^+\hat{U}^+(t)}|0\rangle$$

Now we use $e^{i/\hbar \hat{H}t}\,\hat{a}^+\,e^{-i/\hbar\hat{H}t} = \hat{a}^+ e^{i\omega t}$ (from p. 165 or our discussion of the Heisenberg picture), so (by letting $t \to -t$)

$$\hat{U}(t)\,\hat{a}^+\,\hat{U}^+(t) = e^{-i/\hbar \hat{H}t}\,\hat{a}^+\,e^{i/\hbar\hat{H}t} = \hat{a}^+ e^{-i\omega t}$$

Hence

$$|z(t)\rangle = e^{-i\omega t/2}\, e^{-|z_0 e^{-i\omega t}|^2/2}\, e^{z_0 e^{-i\omega t}\hat{a}^+}|0\rangle$$

$$= e^{-i\omega t/2}\,|z_0 e^{-i\omega t}\rangle$$

This fixes the phase factor λ as $e^{-i\omega t/2}$.

Therefore

$$\langle z_f|U(t,0)|z_i\rangle = \langle z_f|e^{-i\omega t/2}|z_i e^{-i\omega t}\rangle$$

$$= e^{-i\omega t/2}\langle z_f|z_i e^{-i\omega t}\rangle = e^{-i\omega t/2}\, e^{-|z_i|^2/2 - |z_f|^2/2 + z_f^* z_i e^{-i\omega t}}$$

where the last step follows from the result

$$\langle z_1|z_2\rangle = e^{-(|z_1|^2+|z_2|^2)/2 + z_1^* z_2}$$

proved on p. 169.

The coherent state can be decomposed into energy eigenstates as

$$|z\rangle = \sum_{n=0}^{\infty} |n\rangle\langle n|z\rangle = e^{-|z|^2/2} \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!}} |n\rangle$$

The probability $P_n(z)$ of finding a particle in coherent state $|z\rangle$ to have energy $E_n = (n+\frac{1}{2})\hbar\omega$ is thus given by

$$\boxed{P_n(z) = |\langle n|z\rangle|^2 = e^{-|z|^2} \frac{(|z|^2)^n}{n!}} = e^{-\langle n\rangle} \frac{\langle n\rangle^n}{n!}$$

This is the Poisson distribution. The mean value for n is

$\langle n\rangle = |z|^2$; so a coherent state has mean energy

$$\boxed{\langle \hat{H}\rangle = \langle z|\hat{H}|z\rangle = (\langle n\rangle + \tfrac{1}{2})\hbar\omega = (|z|^2 + \tfrac{1}{2})\hbar\omega}$$

The variance of the Poisson distribution is $(\Delta n)^2 = \langle n\rangle$; hence a coherent state has energ uncertainty $\frac{\Delta E}{E} = \frac{\Delta n}{\langle n\rangle} = \frac{1}{\sqrt{\langle n\rangle}} = \frac{1}{|z|} \to 0$ as $|z|^2 \to \infty$, $\langle n\rangle \to \infty$. *)

The coherent states are complete:

$$\boxed{\frac{1}{\pi}\int d^2z\, |z\rangle\langle z| = \frac{1}{\pi}\int d\mathrm{Re}(z)\, d\mathrm{Im}(z)\, |z\rangle\langle z| = \hat{I}}$$

but not orthogonal:

$$\langle z_1|z_2\rangle = e^{-(|z_1|^2+|z_2|^2)/2} \langle 0|e^{z_1^*\hat{a}} e^{z_2\hat{a}^+}|0\rangle$$

$$= e^{-(|z_1|^2+|z_2|^2)/2} \langle 0|e^{z_2\hat{a}^+} \underbrace{\left(e^{-z_2\hat{a}^+} e^{z_1^*\hat{a}} e^{z_2\hat{a}^+}\right)}_{\sum_{n=0}^{\infty}\frac{1}{n!}\left[-z_2\hat{a}^+, e^{z_1^*\hat{a}}\right]_{(n)} = e^{z_2 z_1^*} e^{z_1^*\hat{a}}} |0\rangle$$

$$= e^{-|z_1|^2/2 - |z_2|^2/2 + z_2 z_1^*} \underbrace{\langle 0|e^{z_2\hat{a}^+}}_{\langle 0|} \underbrace{e^{z_1^*\hat{a}}|0\rangle}_{|0\rangle} = e^{-|z_1|^2/2 - |z_2|^2/2 + z_1^* z_2}$$

$$\Rightarrow \boxed{|\langle z_1|z_2\rangle| = e^{-|z_1-z_2|^2}}$$

The distance $|z_1 - z_2|$ in the complex eigenvalue plane ($\mathrm{Re}\, z \sim x$, $\mathrm{Im}\, z \sim p$) determines the degree to which the two eigenstates of $\hat{a}$ are approximately orthogonal.

*) This means that for $\langle n\rangle \to \infty$ (classical limit), the energy uncertainty goes to zero, i.e. the energy of the coherent state becomes well-defined.

$$\frac{1}{\pi}\sum_{nn'}\int d\,Re(z)\, d\,Im(z)\, |n\rangle\langle n|z\rangle\langle z|n'\rangle\langle n'|$$

$$= \frac{1}{\pi}\sum_{n,n'} \int d^2z \; e^{-|z|^2/2} \frac{z^n}{\sqrt{n!}} \; e^{-|z|^2/2} \frac{(z^*)^{n'}}{\sqrt{n'!}} \; |n\rangle\langle n'|$$

$$= \frac{1}{\pi}\sum_{nn'} \frac{1}{\sqrt{n!n'!}} \int_0^\infty |z|\, d|z| \int_{-\pi}^{\pi} d\varphi \; e^{-|z|^2} \, |z|^{n+n'} \; e^{i(n-n')\varphi} \; |n\rangle\langle n'|$$

$$= \frac{2\pi}{\pi} \sum_{nn'} \delta_{nn'} \frac{1}{\sqrt{n!n'!}} \; \frac{1}{2} \underbrace{\int_0^\infty d(|z|^2)\; e^{-|z|^2}(|z|^2)^n}_{\int_0^\infty dt\, t^n e^{-t} \;=\; n!} \; |n\rangle\langle n|$$

$$= \sum_n |n\rangle\langle n| = \hat{1}. \checkmark$$

$$\boxed{|z|^2 = \frac{1}{2}(\xi^2+\zeta^2) = \frac{1}{2}\left(\frac{m\omega}{\hbar}x^2 + \frac{p^2}{m\hbar\omega}\right) = \frac{\frac{p^2}{2m}+\frac{1}{2}m\omega^2x^2}{\hbar\omega}}$$

= energy in units of $\hbar\omega$!