Shankar Chapter 9: The Heisenberg Uncertainty Relations

9.1 Derivation of the uncertainty relation:

Remember the definition of the uncertainty of an observable $\hat{\Omega}$ for a system in state $|\psi\rangle$:

$$\Delta\Omega = \sqrt{\langle\psi|(\hat{\Omega}-\langle\hat{\Omega}\rangle)^2|\psi\rangle}$$

States with $\Delta\Omega = 0$ are eigenstates of $\hat{\Omega}$:

$$\langle\psi|(\hat{\Omega}-\langle\hat{\Omega}\rangle)^2|\psi\rangle = \langle(\hat{\Omega}-\langle\hat{\Omega}\rangle)\psi|(\hat{\Omega}-\langle\hat{\Omega}\rangle)\psi\rangle = 0$$

$$\Rightarrow \hat{\Omega}|\psi\rangle = \langle\hat{\Omega}\rangle|\psi\rangle \quad \Rightarrow |\psi\rangle \text{ eigenstate of } \hat{\Omega} \text{ with eigenvalue } \langle\hat{\Omega}\rangle.$$

All other states have $\Delta\Omega > 0$.

Let $\hat{\Omega}, \hat{\Lambda}$ be two observables (i.e. $\hat{\Omega}^+ = \hat{\Omega}$, $\hat{\Lambda}^+ = \hat{\Lambda}$)

with commutator

$$[\hat{\Omega}, \hat{\Lambda}] = i\hat{\Gamma}$$

(where $\hat{\Gamma} = \hat{\Gamma}^+$ is again Hermitean since $[\hat{\Omega}, \hat{\Lambda}]$ is antihermitean.)

Consider a normalized state $|\psi\rangle$ and compute the uncertainty product

$$(\Delta\Omega)^2(\Delta\Lambda)^2 = \langle\psi|(\hat{\Omega}-\bar{\Omega})^2|\psi\rangle\langle\psi|(\hat{\Lambda}-\bar{\Lambda})^2|\psi\rangle$$

where $\bar{\Omega} = \langle\hat{\Omega}\rangle = \langle\psi|\hat{\Omega}|\psi\rangle$, $\bar{\Lambda} = \langle\hat{\Lambda}\rangle = \langle\psi|\hat{\Lambda}|\psi\rangle$.

For ease of notation introduce $\hat{\omega} \equiv \hat{\Omega} - \bar{\Omega}$, $\hat{\lambda} = \hat{\Lambda} - \bar{\Lambda}$:

$([\hat{\omega}, \hat{\lambda}] = [\hat{\Omega}, \hat{\Lambda}] = i\hat{\Gamma})$

$$(\Delta\Omega)^2 (\Delta\Lambda)^2 = \langle\psi|\hat{\omega}^2|\psi\rangle\langle\psi|\hat{\lambda}^2|\psi\rangle = \langle\hat{\omega}\psi|\hat{\omega}\psi\rangle\langle\hat{\lambda}\psi|\hat{\lambda}\psi\rangle = |\hat{\omega}\psi|^2|\hat{\lambda}\psi|^2$$

since $\hat{\omega} = \hat{\omega}^+$ and $\hat{\lambda} = \hat{\lambda}^+$ ($\hat{\omega}^2 = \hat{\omega}^+\hat{\omega}$ etc.)

Now apply the Schwartz inequality

$$|V_1|^2 |V_2|^2 \geq |\langle V_1|V_2\rangle|^2 \qquad (= \text{ only if } |V_2\rangle = c|V_1\rangle)$$

$$\Rightarrow (\Delta\Omega)^2 (\Delta\Lambda)^2 \geq |\langle\hat{\omega}\psi|\hat{\lambda}\psi\rangle|^2$$

Manipulate r.h.s.:

$$\langle\hat{\omega}\psi|\hat{\lambda}\psi\rangle = \langle\psi|\hat{\omega}^+\hat{\lambda}|\psi\rangle = \langle\psi|\hat{\omega}\hat{\lambda}|\psi\rangle$$

$$\Rightarrow (\Delta\Omega)^2 (\Delta\Lambda)^2 \geq |\langle\psi|\hat{\omega}\hat{\lambda}|\psi\rangle|^2$$

Now $\hat{\omega}\hat{\lambda} = \frac{1}{2}(\hat{\omega}\hat{\lambda} + \hat{\lambda}\hat{\omega}) + \frac{1}{2}(\hat{\omega}\hat{\lambda} - \hat{\lambda}\hat{\omega}) = \frac{1}{2}\underbrace{[\hat{\omega}, \hat{\lambda}]_+}_{\text{anticommutator}} + \frac{1}{2}\underbrace{[\hat{\omega}, \hat{\lambda}]_-}_{\substack{\text{commutator} \\ = \frac{1}{2} i\hat{\Gamma}}}$

Since $[\hat{\omega}, \hat{\lambda}]_-$ is anti-Hermitean, its expectation value is pure imaginary

" $[\hat{\omega}, \hat{\lambda}]_+$ " Hermitean, " " " is real.

Using $|a + ib|^2 = a^2 + b^2$ we get

$$\underbrace{(\Delta\Omega)^2 (\Delta\Lambda)^2}_{} \geq \Big|\underbrace{\tfrac{1}{2}\langle\psi|[\hat{\omega}, \hat{\lambda}]_+|\psi\rangle}_{a} + \underbrace{\tfrac{1}{2}\langle\psi|[\hat{\omega}, \hat{\lambda}]_-|\psi\rangle}_{ib}\Big|^2 =$$

$$= \frac{1}{4}\langle\psi|[\hat{\omega}, \hat{\lambda}]_+|\psi\rangle^2 + \frac{1}{4}\langle\psi|\hat{\Gamma}|\psi\rangle^2 \geq \frac{1}{4}\langle\psi|\hat{\Gamma}|\psi\rangle^2$$

This is the general form of the uncertainty relation.

Special case:

$\hat{\Omega}$ and $\hat{\Lambda}$ are operators that represent canonically conjugate observables, such that $\hat{\Gamma} = \hbar$.

(remember $\{\omega(x,p), \lambda(x,p)\} = 1 \iff [\hat{\Omega}, \hat{\Lambda}] = i\hbar$)

In this case

$$(\Delta\Omega)^2(\Delta\Lambda)^2 \geq \frac{1}{4}\langle\psi|[\hat{\omega},\hat{\lambda}]_+|\psi\rangle^2 + \frac{\hbar^2}{4} \geq \frac{\hbar^2}{4}$$

$$\Rightarrow \boxed{\Delta\Omega \cdot \Delta\Lambda \geq \frac{\hbar}{2}}$$

This inequality becomes an equality if and only if

(i) $\langle\psi|[\hat{\omega},\hat{\lambda}]_+|\psi\rangle = 0$

and

(ii) $\hat{\omega}|\psi\rangle = c\,\hat{\lambda}|\psi\rangle$

9.2. The minimum uncertainty packet

We already showed in Section 7.6 (Physics 827, notes pp. 163 ff.) that states that minimize the uncertainty product for $\hat{X}$ and $\hat{P}$ are Gaussian wave packets. According to what we just derived they solve the equations

$$\left(\hat{P} - \langle\hat{P}\rangle\right)|\psi\rangle = c\left(\hat{X} - \langle\hat{X}\rangle\right)|\psi\rangle \qquad (*)$$

and

$$\langle\psi|\left(\hat{P} - \langle\hat{P}\rangle\right)\left(\hat{X} - \langle\hat{X}\rangle\right) + \left(\hat{X} - \langle\hat{X}\rangle\right)\left(\hat{P} - \langle\hat{P}\rangle\right)|\psi\rangle = 0 \qquad (**)$$

where $\langle\hat{X}\rangle$, $\langle\hat{P}\rangle$, of course, also depend on the solution for $|\psi\rangle$.

Projecting $(*)$ on the x-basis gives the differential equation

$$\left(-i\hbar\frac{d}{dx} - \bar{p}\right)\psi(x) = c\,(x - \bar{x})\,\psi(x) \qquad \text{where } \bar{p} = \langle\psi|\hat{P}|\psi\rangle,\ \bar{x} = \langle\psi|\hat{X}|\psi\rangle$$

with the solution

$$\psi(x) = \psi(0)\, e^{i\bar{p}x/\hbar}\, e^{ic(x-\bar{x})^2/2\hbar}$$

Now let's look at the constraint $(**)$

$$\underbrace{\langle\psi|(\hat{P}-\bar{p})}_{c^*\langle\psi|(\hat{X}-\bar{x})}(\hat{X}-\bar{x}) + (\hat{X}-\bar{x})\underbrace{(\hat{P}-\bar{p})|\psi\rangle}_{c(\hat{X}-\bar{x})|\psi\rangle} = 0 \qquad (\text{using } (*)$$

$$\Rightarrow (c^* + c)\underbrace{\langle\psi|(\hat{X}-\bar{x})^2|\psi\rangle}_{|(\hat{X}-\bar{x})\psi|^2 \geq 0} = 0 \quad \Rightarrow \quad c^* + c = 0 \quad \Rightarrow \quad c = i|c|$$

is pure imaginary

$$\Rightarrow \psi(x) = \psi(0)\, e^{i\bar{p}x/\hbar}\, e^{-\frac{|c|(x-\bar{x})^2}{2\hbar}}$$

Write $\frac{\hbar}{|c|} \equiv \Delta^2$

$$\Rightarrow \boxed{\psi(x) = \psi(0)\, e^{i\bar{p}x/\hbar}\, e^{-\frac{(x-\bar{x})^2}{2\Delta^2}}}$$ as derived before.