Chapter 10: Systems with degrees of freedom

10.1. N particles in one dimension

Start with $N=2$.

Two-particle Hilbert space

Classical description: (x_1, p_1) , (x_2, p_2)

Quantization $\longrightarrow \hat{X}_1, \hat{P}_1 \; ; \; \hat{X}_2, \hat{P}_2$

with

$$\boxed{\begin{aligned} [\hat{X}_i, \hat{P}_j] &= i\hbar \hat{\mathbb{1}} \{x_i, p_j\} = i\hbar\, \delta_{ij}\, \hat{\mathbb{1}} \quad (i,j=1,2) \\ [\hat{X}_i, \hat{X}_j] &= i\hbar \hat{\mathbb{1}} \{x_i, x_j\} = 0 \\ [\hat{P}_i, \hat{P}_j] &= i\hbar \hat{\mathbb{1}} \{p_i, p_j\} = 0 \end{aligned}}$$

In $\hat{X}$-basis: $\{|x_1 x_2\rangle\}$

$$\hat{X}_1 |x_1 x_2\rangle = x_1 |x_1 x_2\rangle , \quad \hat{X}_2 |x_1 x_2\rangle = x_2 |x_1 x_2\rangle$$

Normalization $\langle x_1' x_2' | x_1 x_2\rangle = \delta(x_1 - x_1')\,\delta(x_2 - x_2')$

$$|\psi\rangle \longrightarrow \langle x_1 x_2 | \psi\rangle = \psi(x_1, x_2)$$

$$\left.\begin{aligned} \hat{X}_i &\longrightarrow x_i \\ \hat{P}_i &\longrightarrow -i\hbar \frac{\partial}{\partial x_i} \end{aligned}\right\} i = 1,2$$

$\rho(x_1, x_2) \equiv |\langle x_1 x_2 | \psi\rangle|^2 = |\psi(x_1, x_2)|^2 =$ probability density to find particle 1 near x_1 <u>and</u> particle 2 near x_2, if $|\psi\rangle$ is normalized:

$$\langle\psi|\psi\rangle = 1 = \int_{-\infty}^{\infty} dx_1\, dx_2 \,\langle\psi | x_1 x_2\rangle \langle x_1 x_2 | \psi\rangle$$

$$= \int_{-\infty}^{\infty} dx_1\, dx_2 \; \rho(x_1, x_2)$$

Similar in $\hat{P}$-basis $\{|p_1 p_2\rangle\}$, or any other basis $\{|\omega_1 \omega_2\rangle\}$ of simultaneous eigenstates of commuting operators $\hat{\Omega}_1(\hat{X}_1, \hat{P}_1)$, $\hat{\Omega}_2(\hat{X}_2, \hat{P}_2)$.

Each of these bases spans the 2-particle Hilbert space

$$\mathbb{V}_{1\otimes 2} = \text{span}\{|x_1 x_2\rangle \mid -\infty < x_1, x_2 < \infty\}$$
$$= \text{span}\{|p_1 p_2\rangle \mid -\infty < p_1, p_2 < \infty\}$$
$$= \text{span}\{|\omega_1 \omega_2\rangle \mid \omega_1 \in \{\text{eigenvalues of } \hat{\Omega}_1\}, \omega_2 \in \{\text{eigenvalues of } \hat{\Omega}_2\}\}$$

$\mathbb{V}_{1\otimes 2}$ as a Direct Product Space

- $\hat{X}_1, \hat{P}_1$ with $[\hat{X}_1, \hat{P}_1] = i\hbar \hat{\mathbb{1}}_1$ act on states in the Hilbert space for particle 1:

  $\mathbb{V}_1 = \text{span}\{|x_1\rangle \mid -\infty < x_1 < \infty\} = \text{span}\{|p_1\rangle \mid -\infty < p_1 < \infty\} =$
  $= \text{span}\{|\omega_1\rangle \mid \omega_1 \in \{\text{eigenvalues of } \hat{\Omega}_1(\hat{X}_1, \hat{P}_1)\}\}$

- $\hat{X}_2, \hat{P}_2$ with $[\hat{X}_2, \hat{P}_2] = i\hbar \mathbb{1}_2$ act on states in the Hilbert space $\mathbb{V}_2$ for particle 2.

- $\mathbb{V}_1$ and $\mathbb{V}_2$ are disjoint, completely independent spaces. So an inner product between states $|\psi_1\rangle \in \mathbb{V}_1$ and $|\psi_2\rangle \in \mathbb{V}_2$ is <u>not defined</u>.

- States for the 2-particle system can be defined as follows: Say we measure the positions of particles 1 and 2 as x_1 and x_2:

$$\hat{X}_1|x_1\rangle = x_1|x_1\rangle \quad , \quad \hat{X}_2|x_2\rangle = x_2|x_2\rangle$$

So particle 1 is known to be in eigenstate $|x_1\rangle \in \mathbb{V}_1$, particle 2 is in eigenstate $|x_2\rangle \in \mathbb{V}_2$.

The entire two particle system can be said to be in a state

$\boxed{|x_1\rangle \otimes |x_2\rangle}$ $\longleftrightarrow$ particle 1 at x_1, particle 2 at x_2

- $|x_1\rangle \otimes |x_2\rangle$ is a <u>new object</u>, completely unrelated to the inner product $\langle x_1 | x_2\rangle$ or outer product $|x_1\rangle\langle x_2|$ (where $|x_1\rangle, |x_2\rangle$ are states from the <u>same</u> Hilbert space corresponding to different eigenvalues x_1, x_2)

(For example, $|x\rangle \otimes |x\rangle$ (particle 1 at $x_1 = x$ and particle 2 at $x_2 = x$, too) exists (its probability may be zero, but that's a different matter), whereas $\langle x | x\rangle = \delta(0)$ does not.)

The state $|x_1\rangle \otimes |x_2\rangle \in \mathbb{V}_1 \otimes \mathbb{V}_2$ is called the direct product of $|x_1\rangle \in \mathbb{V}_1$ and $|x_2\rangle \in \mathbb{V}_2$, and the space spanned by these direct product states is called the direct product $\mathbb{V}_1 \otimes \mathbb{V}_2$ of the spaces $\mathbb{V}_1$ and $\mathbb{V}_2$

$$\mathbb{V}_1 \otimes \mathbb{V}_2 = \text{span}\left\{ |x_1\rangle \otimes |x_2\rangle \,\middle|\, -\infty < x_1 < \infty, -\infty < x_2 < \infty \right\}$$

- $\dim(\mathbb{V}_1 \otimes \mathbb{V}_2) = (\dim \mathbb{V}_1) \cdot (\dim \mathbb{V}_2)$

(for each basis vector $|x_1\rangle \in \mathbb{V}_1$ and $|x_2\rangle \in \mathbb{V}_2$ there is one (and only one) basis vector $|x_1\rangle \otimes |x_2\rangle \in \mathbb{V}_1 \otimes \mathbb{V}_2$.)

(Recall: $\dim(\mathbb{V}_1 \oplus \mathbb{V}_2) = (\dim \mathbb{V}_1) + (\dim \mathbb{V}_2)$

for the direct sum of vector spaces $\mathbb{V}_1$ and $\mathbb{V}_2$

if all vectors in $\mathbb{V}_1$ are linearly independent from

all vectors in $\mathbb{V}_2$.

For the direct product there is no such restriction, because (by definition) $\mathbb{V}_1$ and $\mathbb{V}_2$ are completely disjoint Hilbert spaces.)

$\mathbb{V}_1 \otimes \mathbb{V}_2$ is a linear vector space:

$$
\begin{aligned}
\left(\alpha|x_1\rangle + \alpha'|x_1'\rangle\right) \otimes \left(\beta|x_2\rangle + \beta'|x_2'\rangle\right) = \;& \alpha\beta|x_1\rangle \otimes |x_2\rangle \\
&+ \alpha'\beta|x_1'\rangle \otimes |x_2\rangle \\
&+ \alpha\beta'|x_1\rangle \otimes |x_2'\rangle \\
&+ \alpha'\beta'|x_1'\rangle \otimes |x_2'\rangle
\end{aligned}
$$

- While every direct product state $|\varphi_1\rangle \otimes |\varphi_2\rangle$ $(|\varphi_i\rangle \in \mathbb{V}_i)$ is an element of $\mathbb{V}_1 \otimes \mathbb{V}_2$, not every state in $\mathbb{V}_1 \otimes \mathbb{V}_2$ is a direct product state.

Example: $|\psi\rangle = |x_1'\rangle \otimes |x_2'\rangle + |x_1''\rangle \otimes |x_2''\rangle$

cannot be written as $|\varphi_1\rangle \otimes |\varphi_2\rangle$ with $|\varphi_1\rangle \in \mathbb{V}_1$, $|\varphi_2\rangle \in \mathbb{V}_2$

- Inner product of direct product states:

$$
\left(\langle x_1'| \otimes \langle x_2'|\right)\left(|x_1\rangle \otimes |x_2\rangle\right) \equiv \underbrace{\langle x_1'|x_1\rangle}_{\text{inner product in } \mathbb{V}_1} \cdot \underbrace{\langle x_2'|x_2\rangle}_{\text{inner product in } \mathbb{V}_2}
$$

$$
= \delta(x_1 - x_1')\,\delta(x_2 - x_2')
$$

- Action of operators in $\mathbb{V}_1 \otimes \mathbb{V}_2$:

Position of particle 1: $\hat{X}_1^{1\otimes 2}(|x_1\rangle \otimes |x_2\rangle) = (\hat{X}_1 \otimes \mathbb{1}_2)(|x_1\rangle \otimes |x_2\rangle)$

$$= (\hat{X}_1|x_1\rangle) \otimes (\hat{\mathbb{1}}_2|x_2\rangle)$$

Position of particle 2: $\hat{X}_2^{1\otimes 2}(|x_1\rangle \otimes |x_2\rangle) = (\mathbb{1}_1 \otimes \hat{X}_2)(|x_1\rangle \otimes |x_2\rangle)$

$$= \underbrace{(\hat{\mathbb{1}}_1|x_1\rangle)}_{\in \mathbb{V}_1} \otimes \underbrace{(\hat{X}_2|x_2\rangle)}_{\in \mathbb{V}_2}$$

Generally: $\hat{\Omega}_1^{1\otimes 2} \equiv \hat{\Omega}_1 \otimes \mathbb{1}_2$ observable for particle 1

$\hat{\Omega}_2^{1\otimes 2} = \mathbb{1}_1 \otimes \hat{\Omega}_2$ " " " 2

$$\left(\hat{\Gamma}_1 \otimes \hat{\Lambda}_2\right)\left(\underbrace{|\omega_1\rangle}_{\in \mathbb{V}_1} \otimes \underbrace{|\omega_2\rangle}_{\in \mathbb{V}_2}\right) = \underbrace{\left(\hat{\Gamma}_1|\omega_1\rangle\right)}_{\in \mathbb{V}_1} \otimes \underbrace{\left(\hat{\Lambda}_2|\omega_2\rangle\right)}_{\in \mathbb{V}_2}$$

observable for particle 1

observable for particle 2

- Question: can position of particle 1 and momentum of particle 2 be measured simultaneously?

Answer: $\left[\hat{X}_i^{1\otimes 2}, \hat{P}_j^{1\otimes 2}\right] = i\hbar\delta_{ij}\, \hat{\mathbb{1}}_1 \otimes \hat{\mathbb{1}}_2 = i\hbar\, \delta_{ij}\, \hat{\mathbb{1}}_{1\otimes 2}$

$\left[\hat{X}_i^{1\otimes 2}, \hat{X}_j^{1\otimes 2}\right] = 0 = \left[\hat{P}_i^{1\otimes 2}, \hat{P}_j^{1\otimes 2}\right] \quad (i,j = 1,2)$

Yes, they can!

Explicitly:

$$\left[\hat{X}_1^{1\otimes 2}, \hat{P}_2^{1\otimes 2}\right] = \left(\hat{X}_1 \otimes \hat{\mathbb{1}}_2\right)\left(\hat{\mathbb{1}}_1 \otimes \hat{P}_2\right) - \left(\hat{\mathbb{1}}_1 \otimes \hat{P}_2\right)\left(\hat{X}_1 \otimes \hat{\mathbb{1}}_2\right)$$

$$= \left(\hat{X}_1 \hat{\mathbb{1}}_1\right) \otimes \left(\hat{\mathbb{1}}_2 \hat{P}_2\right) - \left(\hat{\mathbb{1}}_1 \hat{X}_1\right) \otimes \left(\hat{P}_2 \hat{\mathbb{1}}_2\right)$$

Exercise 10.1.1 (2)

$$= \hat{X}_1 \otimes \hat{P}_2 - \hat{X}_1 \otimes \hat{P}_2 = 0 \checkmark$$

- Big revelation: $\boxed{\begin{aligned} &|x_1\rangle \otimes |x_2\rangle = |x_1 x_2\rangle \; ! \\ \Rightarrow \; &\mathbb{V}_1 \otimes \mathbb{V}_2 = \mathbb{V}_{1\otimes 2} \end{aligned}}$

i.e. the 2-particle vector space $\mathbb{V}_{1\otimes 2}$ introduced on p. 14 is nothing but the direct product space $\mathbb{V}_1 \otimes \mathbb{V}_2$.

- The notation $|x_1 x_2\rangle$ and $\hat{X}_i |x_1 x_2\rangle = x_i |x_1 x_2\rangle$ $(i=1,2)$ is more compact, so we will usually use it. Context will tell us on which space the operators act. For example, look at the kinetic energy associated with the center of mass motion of a 2-particle system:

$$\hat{T}_{CM} = \frac{\hat{P}_{CM}^2}{2M} = \frac{(\hat{P}_1 + \hat{P}_2)^2}{2(m_1+m_2)} = \frac{\hat{P}_1^2 + 2\overset{[\hat{P}_1,\hat{P}_2]=0}{\hat{P}_1\hat{P}_2} + \hat{P}_2^2}{2M}$$

which is shorthand for

$$2M\hat{T}_{CM}^{1\otimes 2} = (\hat{P}_1 \otimes \hat{\mathbb{1}}_2)^2 + 2\hat{P}_1 \otimes \hat{P}_2 + (\mathbb{1}_1 \otimes \hat{P}_2)^2$$

Basis representation of general vectors in the direct product space

Let $\hat{\Omega}_1$ be an operator on $\mathbb{V}_1$ with nondegenerate eigenfunctions $\psi_{\omega_1}(x_1) \equiv \omega_1(x_1)$ in the coordinate basis.

Similarly, let $\omega_2(x_2)$ represent the corresponding eigenfunctions of the observable $\hat{\Omega}$ for particle 2,

$$\hat{\Omega}_2|\omega_2\rangle = \omega_2|\omega_2\rangle \quad (|\omega_2\rangle \in \mathbb{V}_2) \leftrightarrow \langle x_2|\hat{\Omega}_2|\omega_2\rangle = \omega_2 \langle x_2|\omega_2\rangle = \omega_2\, \omega_2(x_2)$$

Consider as state $|\psi\rangle \in \mathbb{V}_{1\otimes 2}$ with coordinate space wave function $\langle x_1 x_2|\psi\rangle = \psi(x_1, x_2)$.

We can expand $|\psi\rangle$ in the product basis $|\omega_1\rangle \otimes |\omega_2\rangle$:

$$|\psi\rangle = \sum\!\!\!\!\!\!\int_{\omega_1,\omega_2} C_{\omega_1,\omega_2} |\omega_1\rangle \otimes |\omega_2\rangle$$

For the wave function this implies

$$\boxed{\begin{aligned}\psi(x_1,x_2) &= \langle x_1 x_2|\psi\rangle = \left(\langle x_1| \otimes \langle x_2|\right)|\psi\rangle \\ &= \sum\!\!\!\!\!\!\int_{\omega_1\omega_2} C_{\omega_1,\omega_2} \langle x_1|\omega_1\rangle \langle x_2|\omega_2\rangle = \sum\!\!\!\!\!\!\int_{\omega_1\omega_2} C_{\omega_1,\omega_2}\, \omega_1(x_1)\,\omega_2(x_2)\end{aligned}}$$

The state $\hat{P}_1|\psi\rangle$ will be represented by

$$\langle x_1 x_2|\hat{P}_1|\psi\rangle = \sum\!\!\!\!\!\!\int_{\omega_1,\omega_2} C_{\omega_1,\omega_2}\left(\langle x_1|\otimes\langle x_2|\right)\left(\hat{P}_1 \otimes \hat{\mathbb{1}}_2\right)\left(|\omega_1\rangle\otimes|\omega_2\rangle\right)$$

$$= \sum\!\!\!\!\!\!\int_{\omega_1,\omega_2} C_{\omega_1,\omega_2} \langle x_1|\hat{P}_1|\omega_1\rangle \langle x_2|\hat{\mathbb{1}}_2|\omega_2\rangle = \sum\!\!\!\!\!\!\int_{\omega_1,\omega_2} C_{\omega_1,\omega_2}\left(-i\hbar\frac{\partial}{\partial x_1}\omega_1(x_1)\right)\omega_2(x_2)$$

$$= -i\hbar\frac{\partial}{\partial x_1} \sum\!\!\!\!\!\!\int C_{\omega_1,\omega_2}\, \omega_1(x_1)\,\omega_2(x_2) = -i\hbar\frac{\partial}{\partial x_1}\psi(x_1,x_2)$$

From considerations like this follow the simple rules for the x-representation:

$$|\psi\rangle \longleftrightarrow \psi(x_1,x_2)$$
$$\hat{X}_1|\psi\rangle \longleftrightarrow x_1\psi(x_1,x_2)$$
$$\hat{X}_2|\psi\rangle \longleftrightarrow x_2\psi(x_1,x_2)$$
$$\hat{P}_1|\psi\rangle \longleftrightarrow -i\hbar\frac{\partial}{\partial x_1}\psi(x_1,x_2)$$
$$\hat{P}_2|\psi\rangle \longleftrightarrow -i\hbar\frac{\partial}{\partial x_2}\psi(x_1,x_2)$$

You can guess how the analogous rules look in p-representation.

You can also contemplate using the x-representation for particle 1 and the p-representation for particle 2:

$$|\psi\rangle \longleftrightarrow \psi(x_1,p_2) = \langle x_1 p_2|\psi\rangle = \left(\langle x_1|\otimes\langle p_2|\right)|\psi\rangle$$

$$\hat{X}_1|\psi\rangle \longleftrightarrow x_1\,\psi(x_1,p_2)$$
$$\hat{X}_2|\psi\rangle \longleftrightarrow i\hbar\frac{\partial}{\partial p_2}\psi(x_1,p_2)$$
$$\hat{P}_1|\psi\rangle \longleftrightarrow -i\hbar\frac{\partial}{\partial x_1}\psi(x_1,p_2)$$
$$\hat{P}_2|\psi\rangle \longleftrightarrow p_2\psi(x_1,p_2)$$

$$\hat{P}_2\hat{X}_1|\psi\rangle = \hat{X}_1\hat{P}_2|\psi\rangle \longleftrightarrow x_1p_2\psi(x_1,p_2) = p_2x_1\,\psi(x_1,p_2)$$
$$\hat{X}_2\hat{P}_1|\psi\rangle = \hat{P}_1\hat{X}_2|\psi\rangle \longleftrightarrow \hbar^2\frac{\partial}{\partial x_1}\frac{\partial}{\partial p_2}\psi(x_1,x_2) = \hbar^2\frac{\partial}{\partial p_2}\frac{\partial}{\partial x_1}\psi(x_1,p_2)$$