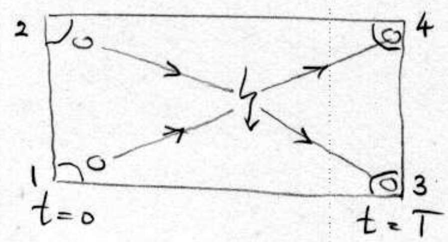
10.3 Identical particles

Classical case:

consider 2 billiard balls as here:



If you exchange balls 1 and 2, the configuration looks exactly the same; still we treat the two configurations as distinct in classical physics. This is because we can follow the trajectories of the particles and thus know that it is ball 2 that ends up at 4 and not ball 1 in the sketch, whereas for the exchanged configuration it would be ball 1 that ends up at hole 4.

The observer who sees everything and disturbs nothing can clearly distinguish between the cases.

Such an observer does not exist in quantum mechanics. We cannot follow the trajectory of the particle in quantum mechanics because we cannot determine its position and velocity at each time simultaneously. Hence we can distinguish the two final states for the two (absolutely identical) balls only by their history, quantum mechanically these final states are <u>indistinguishable</u>!

$\Rightarrow$ two configurations related by the exchange of identical particles must be treated as one and the same and described by the same state vector.

Two-particle systems: symmetric and antisymmetric states.

Let's start from two distinguishable particles 1 and 2 with $x_1 = a$ and $x_2 = b$ (according to a position measurement)

$$\Rightarrow |\psi\rangle = |x_1 = a, x_2 = b\rangle = |ab\rangle \quad \text{just after the measurement}$$

By assumption this can be distinguished from the state

$$|\tilde{\psi}\rangle = |x_1 = b; x_2 = a\rangle = |ba\rangle$$

obtained by exchanging the two particles.

If we repeat the experiment with identical particles, catching one at $x = a$ and the other at $x = b$. So is the state after the measurement $|ab\rangle$ or $|ba\rangle$?

Answer: neither! The state must be unchanged under the exchange of a with b, hence

$$|\psi(a,b)\rangle = \alpha\,|\psi(b,a)\rangle$$

where α is any complex number. ($|\psi\rangle$ and $\alpha|\psi\rangle$ are physically equivalent because the give the same observable matrix elements.). Neither $|\psi(a,b)\rangle = |ab\rangle$ nor $|\psi(a,b)\rangle = |ba\rangle$ satisfies this constraint. However, a linear superposition works: Write $|\psi(a,b)\rangle = \beta|ab\rangle + \gamma|ba\rangle$

and solve

$$\beta|ab\rangle + \gamma|ba\rangle = \alpha\left(\beta|ba\rangle + \gamma|ab\rangle\right)$$

$$\Rightarrow \quad \beta = \alpha\gamma \quad , \quad \gamma = \alpha\beta \qquad \Rightarrow \gamma = \alpha^2\gamma \quad \Rightarrow \boxed{\alpha = \pm 1}$$

So there are two possible state vectors for two identical particles at $x=a$ and $x=b$:

$\alpha=+1$: $\beta|ab\rangle + \gamma|ba\rangle = \beta|ba\rangle + \gamma|ab\rangle$

$|ab\rangle(\beta-\gamma) + |ba\rangle(\gamma-\beta) = 0 \Rightarrow \beta = \gamma = \frac{1}{\sqrt{2}}$ (normalization)

$$\Rightarrow \boxed{|ab, S\rangle = \frac{1}{\sqrt{2}}\left(|ab\rangle + |ba\rangle\right)} \quad \text{(symmetric)}$$

$\alpha=-1$: $\beta|ab\rangle + \gamma|ba\rangle = -\beta|ba\rangle - \gamma|ab\rangle$

$|ab\rangle(\beta+\gamma) + |ba\rangle(\gamma+\beta) = 0 \Rightarrow \beta = -\gamma = \frac{1}{\sqrt{2}}$

$$\Rightarrow \boxed{|ab, A\rangle = \frac{1}{\sqrt{2}}\left(|ab\rangle - |ba\rangle\right)} \quad \text{(antisymmetric)}$$

How to choose between these possibilities?

Bosons and Fermions

Claim: A given species of particles must choose once and for all between S and A states.

Proof by contradiction: suppose the Hilbert space of two identical particles contained both S and A vectors. Then it also contains linear combinations

$$|\psi\rangle = \alpha|\omega_1, \omega_2; S\rangle + \beta|\omega_1', \omega_2'; A\rangle$$

which are neither symmetric nor antisymmetric and thus excluded. So our supposition was wrong.

In nature one finds particles such as pions, photons, gluons that are <u>always</u> in symmetric states and that are called <u>bosons</u>, and other particles such as electrons, protons, neutrons, quarks that are <u>always</u> found in antisymmetric states and are called <u>fermions</u>.
No other possibilities exist in three (or more) dimensions, although in 2 dimensions there exist excitations called <u>anyons</u> that are neither bosons nor fermions and satisfy
$|\psi(a,b)\rangle = e^{i\theta}|\psi(b,a)\rangle$.

Fermions satisfy the Pauli exclusion principle:

$$|\omega\omega, A\rangle = \frac{1}{\sqrt{2}}\left(|\omega\omega\rangle - |\omega\omega\rangle\right) = 0$$

$\Rightarrow$ no two Fermions can occupy the same quantum state $|\omega\rangle$!

Now what determines whether a particle species are bosons or fermions? Answer: their spin. Proof: Spin-Statistics-Theorem (Fierz 1939, Pauli 1940) (beyond the scope of this course)

"integer spin"

Particles with spin $S = m\hbar$, $m \in \mathbb{N}_0$, are bosons
" " " $S = (m+\frac{1}{2})\hbar$, $m \in \mathbb{N}_0$, are fermions

"half-integer spin"

How to construct a Hilbert space for identical bosons or fermions, i.e. a vector space of only symmetric or antisymmetric state vectors?

Let us call the vector space of symmetric (bosonic) vectors $\mathbb{V}_S$, that of antisymmetric (fermionic) states $\mathbb{V}_A$.

How are $\mathbb{V}_S$ and $\mathbb{V}_A$ related to $\mathbb{V}_{1\otimes 2}$? Answer:

$$\mathbb{V}_{1\otimes 2} = \mathbb{V}_S \oplus \mathbb{V}_A$$

Proof: Any linear superposition of (anti)symmetric states is again (anti)symmetric $\Rightarrow$ $\mathbb{V}_{A,S}$ are linear vector spaces.

An arbitrary state $|\psi\rangle \in \mathbb{V}_{1\otimes 2}$ can be decomposed

as $|\psi\rangle = \sum\!\!\!\!\!\!\int_{\omega_1,\omega_2} C_{\omega_1,\omega_2} |\omega_1,\omega_2\rangle$

where $\{|\omega_1\omega_2\rangle = |\omega_1\rangle \otimes |\omega_2\rangle\}$ is a complete set of eigenstates of observables $\hat{\Omega}_1$ for particle 1 and $\hat{\Omega}_2$ for particle 2

$$= \sum\!\!\!\!\!\!\int_{\omega_1,\omega_2} C_{\omega_1,\omega_2}\Big[\tfrac{1}{2}\big(|\omega_1\omega_2\rangle + |\omega_2\omega_1\rangle\big) + \tfrac{1}{2}\big(|\omega_1\omega_2\rangle - |\omega_1\omega_2\rangle\big)$$

$$+ \sum\!\!\!\!\!\!\int_{\omega} C_{\omega\omega}|\omega\omega,S\rangle$$

$$= \frac{1}{\sqrt{2}} \sum\!\!\!\!\!\!\int_{\omega_1\neq\omega_2} C_{\omega_1,\omega_2} |\omega_1,\omega_2,S\rangle + \frac{1}{\sqrt{2}} \sum\!\!\!\!\!\!\int_{\omega_1,\omega_2} C_{\omega_1,\omega_2} |\omega_1,\omega_2,A\rangle \quad (*)$$

$$= \frac{1}{\sqrt{2}}\big(|\psi_S\rangle + |\psi_A\rangle\big)$$

where $|\omega_1,\omega_2,S\rangle = \frac{1}{\sqrt{2}}\big(|\omega_1\omega_2\rangle + |\omega_2\omega_1\rangle\big)$ for $\omega_1 \neq \omega_2$, $|\omega\omega,S\rangle = |\omega\omega\rangle$ for $\omega_1=\omega_2=\omega$, is symmetric and normalized

and $|\omega_1,\omega_2,A\rangle = \frac{1}{\sqrt{2}}\big(|\omega_1\omega_2\rangle - |\omega_2\omega_1\rangle\big)$ " antisymmetric " " ,

and $\langle\omega_1,\omega_2,A|\omega_1,\omega_2,S\rangle = \frac{1}{2}\,\langle\omega_1\omega_2|\omega_1\omega_2\rangle - \langle\omega_1\omega_2|\omega_2\omega_1\rangle + \langle\omega_2\omega_1|\omega_1\omega_2\rangle$
$- \langle\omega_2\omega_1|\omega_2\omega_1\rangle = 0.$

So the states $\{|\omega_1\omega_2, S\rangle\}$ and $\{|\omega_1\omega_2, A\rangle\}$ form bases for orthogonal linear vector spaces $\mathbb{V}_S$ and $\mathbb{V}_A$, respectively, and each state $|\psi\rangle \in \mathbb{V}_{1\otimes 2}$ can be written as

$|\psi\rangle = |\psi_S\rangle + |\psi_A\rangle$ with $\langle\psi_S|\psi_A\rangle = 0$

$$\Rightarrow \mathbb{V}_{1\otimes 2} = \mathbb{V}_S \oplus \mathbb{V}_A .$$

Since the states $|\omega\omega, S\rangle \neq 0$ but $|\omega\omega, A\rangle = 0$, $\mathbb{V}_S$ has (slightly) more states than $\mathbb{V}_A$, so $\dim \mathbb{V}_S > \dim \mathbb{V}_A$.

Now, since $|\omega_1\omega_2, S\rangle = |\omega_2\omega_1, S\rangle$ and $|\omega_1\omega_2, A\rangle = -|\omega_2\omega_1, A\rangle$, the expansion$^{(*)}$ contains each state $|\omega_1 \neq \omega_2, S/A\rangle$ twice.

We can add these terms and simplify the expansion by writing it as

$$\boxed{\begin{aligned}
|\psi_S\rangle &= \sum\!\!\!\!\!\!\!\int_{\omega_2 \leq \omega_1} \sum\!\!\!\!\!\!\!\int_{\omega_1} \tilde{C}_{\omega_1\omega_2} |\omega_1\omega_2, S\rangle \\
|\psi_A\rangle &= \sum\!\!\!\!\!\!\!\int_{\omega_2 < \omega_1} \sum\!\!\!\!\!\!\!\int_{\omega_1} \tilde{C}_{\omega_1\omega_2} |\omega_1\omega_2, A\rangle \\
|\omega_1\omega_2, S/A\rangle &= \frac{1}{\sqrt{2}}\left(|\omega_1\omega_2\rangle \pm |\omega_2\omega_1\rangle\right) \quad \omega_1 \neq \omega_2 \\
|\omega_1\omega_1, S\rangle &= |\omega_1\omega_1\rangle \\
|\omega_1\omega_1, A\rangle &= 0
\end{aligned}}$$

Example What is the probability to find two identical bosonic particles which are in a symmetric state $|\psi_S\rangle$ near positions x_1 and x_2 when we measure their positions? How is this probability density normalized?

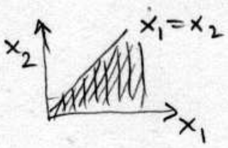
Answer $\quad \rho_S(x_1,x_2) = |\langle x_1 x_2, S|\psi_S\rangle|^2 = \rho_S(x_2,x_1)$

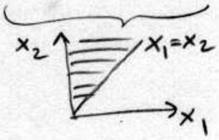
This probability density is normalised as

$$1 = \langle\psi_S|\psi_S\rangle = \langle\psi_S| \underbrace{\underbrace{\int_{-\infty}^{\infty} dx_1 \int_{-\infty}^{x_1} dx_2}_{(x_2 \le x_1)} |x_1 x_2, S\rangle\langle x_1 x_2, S|}_{=\hat{\mathbb{1}} \text{ in the basis of symmetric eigenstates of } \hat{X}_1 \otimes \hat{X}_2.} \psi_S\rangle$$

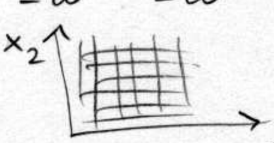
$$= \int_{-\infty}^{\infty} dx_1 \int_{-\infty}^{x_2} dx_2 \, |\langle x_1 x_2, S|\psi_S\rangle|^2 = \int_{-\infty}^{\infty} dx_1 \int_{-\infty}^{x_1} dx_2 \, \rho_S(x_1,x_2)$$

$$= \frac{1}{2}\int_{-\infty}^{\infty} dx_1 \int_{-\infty}^{x_1} dx_2 \, \rho_S(x_1,x_2) + \frac{1}{2}\int_{-\infty}^{\infty} dx_2 \int_{-\infty}^{x_2} dx_1 \, \underbrace{\rho_S(x_2,x_1)}_{=\rho_S(x_1,x_2)}$$





$$= \frac{1}{2}\int_{-\infty}^{\infty} dx_1 \int_{-\infty}^{\infty} dx_2 \, \rho_S(x_1,x_2)$$



If we define $\psi_S(x_1,x_2) \equiv \frac{1}{\sqrt{2}}\langle x_1 x_2, S|\psi_S\rangle$ such that $\int_{-\infty}^{\infty} dx_1 dx_2 |\psi_S(x_1,x_2)|^2 = 1$,

then $\rho_S(x_1,x_2) = 2\,|\psi_S(x_1,x_2)|^2$.

However, if we exploit the symmetry of $|\psi_s\rangle$, we can rewrite

$$\psi_s(x_1,x_2) \equiv \frac{1}{\sqrt{2}}\langle x_1x_2, S|\psi_s\rangle = \frac{1}{2}\Big(\underbrace{\langle x_1x_2|\psi_s\rangle + \langle x_2x_1|\psi_s\rangle}_{\text{Same since } |\psi_s\rangle \text{ is symmetric}}\Big)$$

$$= \langle x_1x_2|\psi_s\rangle$$

and thus

$$1 = \langle\psi_s|\psi_s\rangle = \iint_{-\infty}^{\infty} |\psi_s|^2 dx_1 dx_2 = \iint_{-\infty}^{\infty} \langle\psi_s|x_1x_2\rangle\langle x_1x_2|\psi_s\rangle dx_1 dx_2$$

$$= \langle\psi_s|\underbrace{\iint_{-\infty}^{\infty} dx_1 dx_2 |x_1x_2\rangle\langle x_1x_2|}_{\hat{\mathbb{1}} \text{ in } \mathbb{V}_{1\otimes 2}}|\psi_s\rangle$$

So the normalization condition makes sense both in $\mathbb{V}_s$ and in $\mathbb{V}_{1\otimes 2}$.

<u>Concrete example</u> Suppose we measure the energies of two identical bosons in a box $0 \le x \le L$ and find $n=3$ and $n=4$. The particles are indistinguishable, so the corresponding symmetrized state vector is

$$|\psi_s\rangle = \frac{1}{\sqrt{2}}\big(|3,4\rangle + |4,3\rangle\big)$$

(we label the 2-particle states as $|n_1 n_2\rangle$.)

The wavefunction in x-representation is

$$\psi_s(x_1x_2) = \frac{1}{\sqrt{2}}\langle x_1x_2, S|\psi_s\rangle = \frac{1}{2}\big(\langle x_1x_2| + \langle x_2x_1|\big)\Big(\frac{1}{\sqrt{2}}(|3,4\rangle + |4,3\rangle)\Big) =$$

$$= \frac{1}{2\sqrt{2}}\Big(\langle x_1 x_2 | 34 \rangle + \langle x_1 x_2 | 43 \rangle + \langle x_2 x_1 | 34 \rangle + \langle x_2 x_1 | 43 \rangle \Big)$$

$$= \frac{1}{2\sqrt{2}}\Big(\underbrace{\langle x_1 | 3 \rangle \langle x_2 | 4 \rangle}_{\psi_3(x_1)\,\psi_4(x_2)} + \underbrace{\langle x_1 | 4 \rangle \langle x_2 | 3 \rangle}_{\psi_4(x_1)\,\psi_3(x_2)} + \underbrace{\langle x_2 | 3 \rangle \langle x_1 | 4 \rangle}_{\psi_3(x_2)\,\psi_4(x_1)} + \underbrace{\langle x_2 | 4 \rangle \langle x_1 | 3 \rangle}_{\psi_4(x_2)\,\psi_3(x)} \Big)$$

$$= \frac{1}{\sqrt{2}}\Big(\psi_3(x_1)\,\psi_4(x_2) + \psi_4(x_1)\,\psi_3(x_2) \Big) = \langle x_1 x_2 | \psi_S \rangle$$

(here $\psi_n(x) = \langle x | n \rangle = \sqrt{\frac{2}{L}}\sin\!\left(n\pi \frac{x}{L}\right)$)

For fermions one simply replaces $\psi_S(x_1, x_2)$ by

$$\psi_A(x_1 x_2) \equiv \frac{1}{\sqrt{2}} \langle x_1 x_2, A | \psi_A \rangle = \langle x_1 x_2 | \psi_A \rangle$$

and finds

$$\rho_A(x_1 x_2) = 2\,|\psi_A(x_1 x_2)|^2$$

with normalisation

$$1 = \underbrace{\int_{-\infty}^{\infty} dx_1 \int_{-\infty}^{x_1} dx_2}_{x_2 < x_1}\, \rho_A(x_1 x_2) = \frac{1}{2}\iint_{-\infty}^{\infty} dx_1\, dx_2\, \rho_A(x_1 x_2)$$

$$= \iint_{-\infty}^{\infty} dx_1\, dx_2\, |\psi_A(x_1 x_2)|^2.$$

For the concrete example above ($n=3$ and $n=4$ in a box) one has

$$|\psi_A\rangle = \frac{1}{\sqrt{2}}\big(|3,4\rangle - |4,3\rangle \big)$$

and finds

$$\psi_A(x_1 x_2) = \langle x_1 x_2 | \psi_A \rangle = \frac{1}{\sqrt{2}} \langle x_1 x_2, A | \psi_A \rangle$$

$$= \frac{1}{\sqrt{2}} \left(\psi_3(x_1)\psi_4(x_2) - \psi_4(x_1)\psi_3(x_2) \right)$$

$$= \frac{1}{\sqrt{2}} \begin{vmatrix} \psi_3(x_1) & \psi_4(x_1) \\ \psi_3(x_2) & \psi_4(x_2) \end{vmatrix}$$

In general, if $|\psi_A\rangle = \frac{1}{\sqrt{2}} \left(|\omega_1 \omega_2\rangle - |\omega_2 \omega_1\rangle \right)$, then

$$\psi_A(x_1 x_2) = \frac{1}{\sqrt{2}} \begin{vmatrix} \psi_{\omega_1}(x_1) & \psi_{\omega_2}(x_1) \\ \psi_{\omega_1}(x_2) & \psi_{\omega_2}(x_2) \end{vmatrix} = \frac{1}{\sqrt{2}} \begin{array}{c} \begin{vmatrix} \langle \omega_1 | x_1 \rangle & \langle \omega_2 | x_1 \rangle \\ \langle \omega_1 | x_2 \rangle & \langle \omega_2 | x_2 \rangle \end{vmatrix} \begin{array}{l} \leftarrow x_1 \\ \leftarrow x_2 \end{array} \\ \begin{array}{cc} \uparrow \quad & \quad \uparrow \\ \omega_1 \quad & \quad \omega_2 \end{array} \end{array}$$

This determinant is known as

<u>Slater determinant.</u>