How can we find out experimentally whether a given species of particle is a boson or fermion?

Take two identical particles and put them (to simplify the argument) in a 1-d box. Let us assume the box is large enough that on average the potential energy due to interaction between the particles can be ignored relative to their kinetic energy. Make an energy measurement, and say, we find one in the state $n=3$ and the other in the state $n=4$. Depending on the statistics of the particles, the probability density in position would be

$$\rho_{S/A}(x_1,x_2) = 2\left|\psi_{S/A}(x_1,x_2)\right|^2 = 2\left|\frac{1}{\sqrt{2}}\left(\psi_3(x_1)\psi_4(x_2) \pm \psi_4(x_1)\psi_3(x_2)\right)\right|^2$$

$$= |\psi_3(x_1)|^2\,|\psi_4(x_2)|^2 + |\psi_4(x_1)|^2\,|\psi_3(x_2)|^2$$

$$\pm\left(\psi_3^*(x_1)\psi_4(x_1)\psi_4^*(x_1)\psi_3(x_2) + \psi_4^*(x_1)\psi_3(x_1)\psi_3^*(x_2)\psi_4(x_2)\right)$$

The first two terms describe the probability density for finding the particle with energy E_3 near x_1 <u>and</u> the particle with E_4 near x_2, <u>or</u> (vice versa) the particle with E_3 near x_2 <u>and</u> at the same time the particle with E_4 near x_1. This is what we would expect for distinguishable particles. The $\pm$ terms (called interference terms) arise from the (anti-) symmetrization of the 2-particle state and distinguish between bosons and fermions by their sign.

The effect is most dramatic when considering the probability for both particles at the same position x (this makes sense for photons which do not interact with each other, but for photons we should also account for their helicity which we here ignore):

$$\rho_D(x,x) = 2\,|\psi_3(x)|^2\,|\psi_4(x)|^2 \qquad \text{distinguishable particles}$$

$$\rho_S(x,x) = 4\,|\psi_3(x)|^2\,|\psi_4(x)|^2 \qquad \text{bosons !}$$

$$\rho_A(x,x) = 0 \qquad \text{fermions!}$$

So, compared to classical distinguishable particles, bosons like to clump together while fermions avoid each other, even if they do not repel each other by any interaction!

So this type of measurement allows us to decide whether (a) the particles are identical or distinguishable, and (b) fermions or bosons.

More than 2 identical particles

Illustrate the general case with case of 3 non-interacting particles in a 1-dimensional box:

$$E = \frac{\hbar^2 \pi^2}{2mL^2} \left(n_1^2 + n_2^2 + n_3^2 \right)$$

We have $3! = 6$ product states from which to construct symmetric and antisymmetric states:

$$|n_1 n_2 n_3\rangle,\ |n_1 n_3 n_2\rangle,\ |n_2 n_3 n_1\rangle,\ |n_2 n_1 n_3\rangle,\ |n_3 n_1 n_2\rangle,\ |n_3 n_2 n_1\rangle$$

We write

$$|\psi(n_1 n_2 n_3)\rangle = \beta_1 |n_1 n_2 n_3\rangle + \beta_2 |n_1 n_3 n_2\rangle + \dots + \beta_6 |n_3 n_2 n_1\rangle$$

and demand

$$|\psi(P(n_1 n_2 n_3))\rangle = \alpha_p |\psi(n_1 n_2 n_3)\rangle$$

for any permutation $P(n_1 n_2 n_3)$ that exchanges any two particles. Since for such an exchange $P^2 = \mathbb{1}$, we have

$\alpha_p^2 = 1 \Rightarrow \alpha_p = \pm 1$ for any two-particle exchange.

A little exercise shows that there are only two possible states with this property:

$$|n_1 n_2 n_3; S\rangle = \frac{1}{\sqrt{3!}} \Big[|n_1 n_2 n_3\rangle + |n_1 n_3 n_2\rangle + |n_2 n_3 n_1\rangle + |n_2 n_1 n_3\rangle + |n_3 n_1 n_2\rangle + |n_3 n_2 n_1\rangle \Big] \quad \text{(bosons)}$$

$$|n_1 n_2 n_3; A\rangle = \frac{1}{\sqrt{3!}} \Big[|n_1 n_2 n_3\rangle - |n_1 n_3 n_2\rangle + |n_2 n_3 n_1\rangle - |n_2 n_1 n_3\rangle + |n_3 n_1 n_2\rangle - |n_3 n_2 n_1\rangle \Big] \quad \text{(fermions)}$$

For bosons
(This normalization holds only if $n_1 \neq n_2 \neq n_3$; for fermions the state vanishes unless $n_1 \neq n_2 \neq n_3$.)

The x-space wavefunction is in general

$$\psi_{S/A}(x_1 x_2 x_3) = \frac{1}{\sqrt{3!}} \langle x_1 x_2 x_3 ; S/A | \psi_{S/A} \rangle = \langle x_1 x_2 x_3 | \psi_{S/A} \rangle$$

with $\int_{-\infty}^{\infty} dx_1 \int_{-\infty}^{\infty} dx_2 \int_{-\infty}^{\infty} dx_3 |\psi_{S/A}(x_1 x_2 x_3)|^2 = 1.$

For the 3 particles in the box

$$\psi_{n_1 n_2 n_3, S/A}(x_1 x_2 x_3) = \frac{1}{\sqrt{3!}} \Big[\psi_{n_1}(x_1)\psi_{n_2}(x_2)\psi_{n_3}(x_3) \pm \psi_{n_1}(x_1)\psi_{n_3}(x_2)\psi_{n_2}(x_3)$$

$$+ \psi_{n_2}(x_1)\psi_{n_3}(x_2)\psi_{n_1}(x_3) \pm \psi_{n_2}(x_1)\psi_{n_1}(x_2)\psi_{n_3}(x_2)$$

$$+ \psi_{n_3}(x_1)\psi_{n_1}(x_2)\psi_{n_2}(x_3) \pm \psi_{n_3}(x_1)\psi_{n_2}(x_2)\psi_{n_1}(x_3) \Big]$$

For the fermionic case we have again the Slater-determinant representation

$$\psi_{n_1 n_2 n_3, A}(x_1 x_2 x_3) = \frac{1}{\sqrt{3!}} \begin{vmatrix} \psi_{n_1}(x_1) & \psi_{n_2}(x_1) & \psi_{n_3}(x_1) \\ \psi_{n_1}(x_2) & \psi_{n_2}(x_2) & \psi_{n_3}(x_2) \\ \psi_{n_1}(x_3) & \psi_{n_2}(x_3) & \psi_{n_3}(x_3) \end{vmatrix}$$

In general

$$\boxed{\psi_{n_1 \dots n_n, S}(x_1 x_2 \dots x_n) = \frac{1}{\sqrt{n!}} \sum_{\sigma \in S_n} \psi_{\sigma(n_1)}(x_1)\psi_{\sigma(n_2)}(x_2) \dots \psi_{\sigma(n_n)}(x_n)}$$

where S_n is the group of permutations of $(123 \dots n)$, and the sum is over all permutations $\sigma(12 \dots n) = (\sigma(1)\sigma(2) \dots \sigma(n))$

For antisymmetric states we define the sign of a permutation

$$\text{sign}(\sigma) = \begin{cases} +1 & \sigma = \text{even number of pairwise exchanges} \\ -1 & \sigma = \text{odd number of pairwise exchanges} \end{cases}$$

$$\Rightarrow \boxed{\psi_{n_1 n_2 \dots n_n, A}(x_1 x_2 \dots x_n) = \frac{1}{\sqrt{n!}} \sum_{\sigma \in S_n} \text{sign}(\sigma) \psi_{\sigma(n_1)}(x_1) \dots \psi_{\sigma(n_n)}(x_n)}$$

Note that all states must be completely symmetric or completely antisymmetric. However, the vector space $\mathbb{V}_1 \otimes \mathbb{V}_2 \otimes \ldots \otimes \mathbb{V}_n$ contains many more vectors than can be constructed from linear superpositions of completely symmetrized or antisymmetrized product states: For N particles, we have for each set of N different eigenvalues $N!$ product states made from the 1-particle eigenstates, but out of these $N!$ states only 2 are either completely symmetric or completely antisymmetric.

Hence for $N \geq 3$, $\mathbb{V}_1 \otimes \mathbb{V}_2 \otimes \ldots \otimes \mathbb{V}_N \neq \mathbb{V}_S \oplus \mathbb{V}_A$

and $\dim(\mathbb{V}_S \oplus \mathbb{V}_A) < \dim(\mathbb{V}_{1 \otimes 2 \otimes \ldots \otimes N})$.