When can we ignore (anti-)symmetrization?

When we describe any system, we always make the assumption that we can study it in isolation from the rest of the world. This never completely exact (due to long-range gravitational interactions), off often an excellent approximation:

$$\mathcal{H}_{universe} = \mathcal{H}_{sys} + \mathcal{H}_{rest} = \mathcal{H}_{sys}(\vec{x}_s, \vec{p}_s) + \mathcal{H}_{rest}(\vec{x}_r, \vec{p}_r)$$

without interaction that depends on both $\vec{x}_s$ and $\vec{x}_r$.

Quantum mechanically, the separability of $\hat{H} = \hat{H}_{sys} + \hat{H}_{rest}$ leads to factorization of the wave function of the universe:

$$\psi_{universe}(x_s, x_r) = \psi_{sys}(x_s)\,\psi_{rest}(x_r)$$

Such that the probability to find the system at x_s, without caring about the rest of the system, is

$$P(x_s) = \int |\psi_{universe}(x_s, x_r)|^2 dx_r = |\psi_{sys}(x_s)|^2 \underbrace{\int |\psi_{rest}(x_r)|^2 dx_r}_{=1}$$

$$= |\psi_{sys}(x_s)|^2$$

This is what allows to ignore the wavefunction ψ_{rest}.

? But what if the wavefunction of the universe contains identical particles in both the system and the rest?

Even without interaction, symmetrization of the wavefunction destroys the factorization of probabilities, as we saw in the case of two particles:

$$P_{S/A}(x_s, x_r) \neq P(x_s)\,P(x_r) \qquad ↯$$

$$\Rightarrow \; P_{S/A}(x_s) = \int P_{S/A}(x_s, x_r)\,dx_r \;\neq P(x_s) = |\psi(x_s)|^2$$

In this case we cannot ignore the rest of the universe, which makes it very hard to calculate anything!

However, in practical cases this not an issue. For localized wavefunctions, symmetrization can be ignored between particles that are separated by distances much larger than their (finite) position uncertainty, even if they are identical.

Consider neutral pions which have spin 0 and are bosons. Suppose we find one pion trapped in the ground state of some oscillator potential on the earth, and another trapped similarly on the moon. We know that the respective 1-particle wave functions are Gaussians

$$G_E(x_1) \sim e^{-\frac{m\omega}{2\hbar}(x_1 - x_E)^2}, \quad G_M(x_2) \sim e^{-\frac{m\omega}{2\hbar}(x_2 - x_M)^2}$$

where x_E is a point on earth and x_M a point on the moon.

The symmetrized wavefunction for these 2 pions is

$$\psi_s(x_1 x_2) = \frac{1}{\sqrt{2!}}\Big(G_E(x_1)\, G_M(x_2) + G_M(x_1)\, G_E(x_2)\Big)$$

The probability density for finding one pion near x_1 and the other near x_2 is (see p. 37)

$$\rho(x_1 x_2) = 2|\psi_s|^2 = |G_E(x_1)|^2 |G_M(x_2)|^2 + |G_M(x_1)|^2 |G_E(x_2)|^2$$
$$+ G_E^*(x_1)\, G_M(x_1)\, G_M^*(x_2) G_E(x_2) + G_M^*(x_1) G_E(x_1) G_E^*(x_2) G_M(x_2)$$

What is the probability to find one of the pions at a point y_E on earth, irrespective of where the other pion is?

Since $\rho(x_1, x_2) = \rho(x_2, x_1)$, we can set w.o.l.o.g. $x_1 = y_E$ and integrate over x_2:

$$\rho(y_E) = |G_E(y_E)|^2 \underbrace{\int dx_2\, |G_M(x_2)|^2}_{1} + \underbrace{|G_M(y_E)|^2}_{\substack{\sim e^{-\frac{m\omega}{\hbar}(y_E - x_M)^2} \\ \lll 1}} \underbrace{\int |G_E(x_2)|^2 dx_2}_{1}$$

$$+ G_E^*(y_E) \underbrace{G_M(y_E)}_{\sim e^{-\frac{m\omega}{2\hbar}(y_E - x_M)^2} \lll 1} \int G_M^*(x_2)\, G_E(x_2)\, dx_2$$

$$+ \underbrace{G_M^*(y_E)}_{\sim e^{-\frac{m\omega}{2\hbar}(y_E - x_M)^2} \lll 1} G_E(y_E) \int G_E^*(x_2)\, G_M(x_2)\, dx_2$$

Due to $(y_E - x_M)^2 \gg \frac{2\hbar}{m\omega} = (\text{classical turning point})^2$, the last three terms are exceedingly small, and we obtain

$$\rho(y_E) = |G_E(y_E)|^2 \sim e^{-\frac{m\omega}{\hbar}(y_E - x_E)^2}$$

as if the second pion did not exist! (Same holds for fermions.)

This would not work if the two pions were not separated by much more than the widths of their Gaussians. If they were Gaussians without an oscillator potential to trap them, their widths would grow, and eventually, as the wavepackets flow into each other, the separation of "system pion" and "rest pion" breaks down.

Consider 3 pions. We know that two pions must be in a symmetrized state, but couldn't 3 pion be instead in an antisymmetrized state? The answer is: no! Take two of the pions to be on earth and the third on the moon. In this case we showed that we can approximate the 3-pion wavefunction exceedingly well as

$$(*) \quad \psi(x_1, x_2, x_3) \cong \psi_E(x_1, x_2)\, \psi_M(x_3)$$

where ψ_E is either symmetric or antisymmetric

Integrating over the moon pion we get

$$\rho(x_1 x_2) = 2\,|\psi_E(x_1 x_2)|^2$$

This must be symmetric, hence ψ_E must be symmetric, hence $\psi(x_1 x_2 x_3)$ must also have been completely symmetric before we approximated it in (*).