Example $\hat{R} \equiv \hat{R}(\frac{\pi}{2}\vec{e}_1)$

$$\begin{aligned} \hat{R}\,|1\rangle &= |1\rangle \\ \hat{R}\,|2\rangle &= |3\rangle \\ \hat{R}\,|3\rangle &= |-2\rangle \end{aligned} \quad \Rightarrow \quad [R_{ij}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$$

Exercise: How can you describe the action of
(Homework) $\hat{\Omega}$ whose matrix elements in the same basis are given by $\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$?

Matrix forms of some specific operators

(1) Identity operator: $I_{ij} = \langle i|\hat{I}|j\rangle = \langle i|j\rangle = \delta_{ij} \Leftrightarrow \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix}$

(2) Projection operators:

Consider the expansion of an arbitrary ket $|V\rangle$ as

$$|V\rangle = \sum_{i=1}^{n} |i\rangle\langle i|V\rangle = \left(\sum_{i=1}^{n} |i\rangle\langle i|\right)|V\rangle$$

Since this is true for all $|V\rangle$, the expression in brackets must be the identity operator:

$$\hat{I} = \sum_{i=1}^{n} |i\rangle\langle i| \equiv \sum_{i=1}^{n} \hat{\mathbb{P}}_i \qquad (*)$$

The object $\hat{\mathbb{P}}_i = |i\rangle\langle i|$ is called the projection operator for the ket $|i\rangle$. Eq. $(*)$ is called completeness relation; it expresses the identity operator as a sum over projection operators. This will prove very valuable.

Now consider

$$\hat{\mathbb{P}}_i |V\rangle = |i\rangle\langle i|V\rangle = v_i |i\rangle$$

Clearly, $\hat{\mathbb{P}}_i$ is linear. Whatever $|V\rangle$, $\hat{\mathbb{P}}_i|V\rangle$ points in direction of $|i\rangle$. $\hat{\mathbb{P}}_i$ projects out the component of $|V\rangle$ along $|i\rangle$. The completeness relation says, that the sum of the projections of $|V\rangle$ along the n basis directions reproduces the vector $|V\rangle$.

$\hat{\mathbb{P}}_i$ can also act on bras:

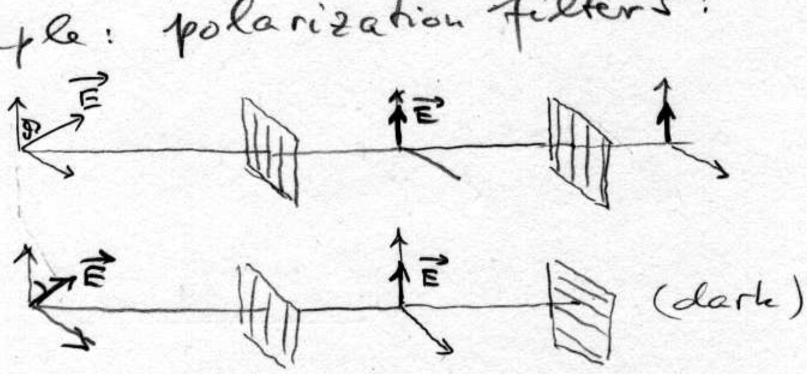
$$\langle V|\hat{\mathbb{P}}_i = \langle V|i\rangle\langle i| = \langle i| v_i^*$$

We have

$$\hat{\mathbb{P}}_i \hat{\mathbb{P}}_j = |i\rangle\langle i|j\rangle\langle j| = \delta_{ij} |j\rangle\langle j| = \delta_{ij} \hat{\mathbb{P}}_j$$

Projectors along orthogonal directions give 0, projecting multiple times along the same direction reproduces the result of the first projection.

Example: polarization filters:



The matrix elements of $\hat{\mathbb{P}}_i$ are

$$(\mathbb{P}_i)_{kl} = \langle k|i\rangle\langle i|l\rangle = \delta_{ki}\delta_{li}$$

$$= \begin{array}{cc} \begin{bmatrix} 0 & \cdots & \cdots & \cdots & 0 \\ \vdots & \ddots & & & \vdots \\ \vdots & 0 & 1 & 0 & \vdots \\ \vdots & & & \ddots & \vdots \\ 0 & \cdots & \cdots & \cdots & 0 \end{bmatrix} & \leftarrow i \\ \uparrow & \\ i & \end{array}$$

$\Rightarrow$ only one non-zero matrix element of value 1 on the diagonal at position i.

Adding all projection operators fills the diagonal with 1's and thus reproduces the identity operator.

(3) Matrix elements of products of operators:

$$(\Omega\Lambda)_{ij} = \langle i|\hat{\Omega}\hat{\Lambda}|j\rangle = \langle i|\hat{\Omega}\,\hat{I}\,\hat{\Lambda}|j\rangle$$

$$= \langle i|\hat{\Omega}\left(\sum_{k=1}^{n}|k\rangle\langle k|\right)\hat{\Lambda}|j\rangle$$

$$= \sum_{k=1}^{n}\langle i|\hat{\Omega}|k\rangle\langle k|\hat{\Lambda}|j\rangle = \sum_{k=1}^{n}\Omega_{ik}\Lambda_{kj}$$

matrix multiplication!

(4) Adjoint of an operator:

For a ket $\alpha|V\rangle = |\alpha V\rangle$, the corresponding bra is

$\langle\alpha V| = \langle V|\alpha^*$ (not $\langle V|a$!!)

Similarly, for a ket $\hat{\Omega}|V\rangle = |\hat{\Omega}V\rangle$, the corresponding bra is

$$\langle \hat{\Omega} V| = \langle V|\hat{\Omega}^+$$

which defines the adjoint operator $\hat{\Omega}^+$.

As $\hat{\Omega}$ turns $|V\rangle$ in $|V'\rangle = |\hat{\Omega}V\rangle$, $\hat{\Omega}^+$ turns $\langle V|$ into $\langle V'| = \langle V|\hat{\Omega}^+$.

Its matrix elements are

$$(\Omega^+)_{ij} = \langle i|\hat{\Omega}^+|j\rangle = \langle \Omega i|j\rangle = \langle j|\Omega i\rangle^*$$
$$= \langle j|\Omega|i\rangle^* = \Omega_{ji}^*$$

So the matrix for $\hat{\Omega}^+$ is obtained from that for $\hat{\Omega}$ by transposing columns and rows and complex conjugating each matrix element.

Adjoint of a product of operators:

$$(\hat{\Omega}\hat{\Lambda})^+ = \Lambda^+ \Omega^+$$

(Proof: $\langle V|(\hat{\Omega}\hat{\Lambda})^+ = \langle(\hat{\Omega}\hat{\Lambda})V| = \langle \hat{\Omega}(\hat{\Lambda}V)| = \langle \hat{\Lambda}V|\Omega^+$
$= (\langle V|\hat{\Lambda}^+)\hat{\Omega}^+ = \langle V|\hat{\Lambda}^+\hat{\Omega}^+$)

Example: consider $\alpha_1|V_1\rangle = \alpha_2|V_2\rangle + \alpha_3|V_3\rangle\langle V_4|V_5\rangle + \alpha_4\hat{\Omega}\hat{\Lambda}|V_6\rangle$

What is the adjoint?

Answer: $\langle V_1|\alpha_1^* = \langle V_2|\alpha_2^* + \langle V_5|V_4\rangle\langle V_3|\alpha_3^* + \langle\hat{\Omega}\hat{\Lambda}V_6|\alpha_4^*$

$= \langle V_2|\alpha_2^* + \langle V_3|\alpha_3^*\langle V_4|V_5\rangle^* + \langle V_6|\hat{\Lambda}^+\Omega^+\alpha_4^*$

Hermitian, Anti-Hermitian, and Unitary Operators

Defn: A Hermitian operator is self-adjoint: $\hat{\Omega}^+ = \hat{\Omega}$

Defn: An anti-Hermitian operator satisfies $\hat{\Omega}^+ = -\hat{\Omega}$

Defn: A unitary operator satisfies $\hat{U}\hat{U}^+ = \hat{I}$,

i.e. $\hat{U}^+ = \hat{U}^{-1}$

Since the inverse satisfies $\hat{U}^{-1}\hat{U} = \hat{U}\hat{U}^{-1} = \hat{I}$, we also have $\hat{U}^+\hat{U} = \hat{I}$. (This holds in finite dimensional vector spaces, but may not hold in infinite dimensional vector spaces without additional restrictions.)

- Any operator can be decomposed into its Hermitian and anti-Hermitian parts:

$$\hat{\Omega} = \underbrace{\frac{\hat{\Omega}+\hat{\Omega}^+}{2}}_{\text{Hermitian}} + \underbrace{\frac{\hat{\Omega}-\hat{\Omega}^+}{2}}_{\text{anti-Hermitian}}$$

- Any product of unitary operators is unitary.

$$(\hat{U}_1\hat{U}_2)^+ = \hat{U}_2^+\hat{U}_1^+ \quad \Rightarrow \quad (\hat{U}_1\hat{U}_2)^+(\hat{U}_1\hat{U}_2) = \hat{U}_2^+\underbrace{\hat{U}_1^+\hat{U}_1}_{\hat{I}}\hat{U}_2$$
$$= \hat{U}_2^+\hat{U}_2 = \hat{I}$$

- Unitary operators preserve inner product:

$$|V_1'\rangle = \hat{U}|V_1\rangle \,;\; |V_2'\rangle = U|V_2\rangle \Rightarrow \langle V_2'|V_1'\rangle = \langle \hat{U}V_2|\hat{U}V_1\rangle$$
$$= \langle V_2|\underbrace{\hat{U}^+\hat{U}}_{\hat{I}}|V_1\rangle = \langle V_2|V_1\rangle$$

Theorem Both the columns of an $n \times n$ unitary matrix and the rows of such a matrix form orthonormal basis sets of dimension n.

Proof: $\delta_{ij} = \langle i|\hat{I}|j\rangle = \langle i|\hat{U}^+\hat{U}|j\rangle$

$$= \sum_k \langle i|\hat{U}^+|k\rangle\langle k|\hat{U}|j\rangle$$

$$= \sum_k (U^+)_{ik} U_{kj} = \sum_k U^*_{ki} U_{kj}$$

This proves the result for the columns. (U_{kj} are the elements of the j^{th} column.)

The proof for the rows follows similarly after substituting $\hat{I} = \hat{U}\hat{U}^+$.

I.7 Unitary transformations of operators

Under a unitary transformation

$$|V\rangle \rightarrow \hat{U}|V\rangle$$

the matrix elements of an operator change as

$$\langle V'|\hat{\Omega}|V\rangle \longrightarrow \langle UV'|\hat{\Omega}|\hat{U}V\rangle = \langle V|\hat{U}^+\hat{\Omega}\hat{U}|V\rangle$$

So instead of transforming the "states" $|V\rangle \rightarrow \hat{U}|V\rangle$ we can transform the operator $\hat{\Omega} \rightarrow \hat{U}^+\hat{\Omega}\hat{U}$

Since we leave the vectors alone and transform only the operators, this is called a passive transformation.

I.8. The eigenvalue problem

For each operator $\hat{\Omega}$, there are certain kets that are simply rescaled (i.e. multiplied by a constant) when $\hat{\Omega}$ acts on them:

$$\hat{\Omega}|V\rangle = \omega|V\rangle \qquad (*)$$

Any ket $|V\rangle$ with that property is called an <u>eigenket</u> of $\hat{\Omega}$, and ω is called the <u>eigenvalue</u> of $\hat{\Omega}$ for that ket. $(*)$ is called an <u>eigenvalue equation</u>.

<u>Example</u>; Consider $\hat{\Omega} = \hat{I}$

since $\hat{I}|V\rangle = |V\rangle$

all vectors are eigenvalues of $\hat{I}$, and 1 is the only eigenvalue.

<u>Example</u>: Consider $\hat{\Omega} = \hat{\mathbb{P}}_V$ where $|V\rangle$ is normalized:

$\hat{\mathbb{P}}_V = |V\rangle\langle V|$

(1) Any ket $|\alpha V\rangle$ (parallel to $|V\rangle$) is an eigenket with eigenvalue 1: $\hat{\mathbb{P}}_V|\alpha V\rangle = \alpha|V\rangle\underbrace{\langle V|V\rangle}_{1}$

$= |\alpha V\rangle$

(2) Any ket $|V_\perp\rangle$ perpendicular to $|V\rangle$ is an eigenket with eigenvalue 0: $\hat{\mathbb{P}}_V|V_\perp\rangle = |V\rangle\underbrace{\langle V|V_\perp\rangle}_{0} = 0$

$= 0|V_\perp\rangle$

(3) Any other ket (neither parallel nor perpendicular) is not an eigenket:

$$\hat{\mathbb{P}}_V\left(\alpha|V\rangle + \beta|V_\perp\rangle\right) = \alpha|V\rangle \neq \gamma\left(\alpha|V\rangle + \beta|V_\perp\rangle\right)$$

A systematic approach to finding all eigenvalues and eigenvectors of an operator:

The characteristic equation

Let's rewrite the eigenvalue equation as

$$(\hat{\Omega} - \omega \hat{I})|V\rangle = |0\rangle \qquad (*)$$

If $(\hat{\Omega} - \omega\hat{I})^{-1}$ exists, we can operate with it on both sides to get

$$|V\rangle = (\hat{\Omega} - \omega \hat{I})^{-1}|0\rangle$$

But this makes no sense: any finite operator (with finite matrix elements) maps the null vector onto itself. Hence the assumption that $(\hat{\Omega} - \omega\hat{I})^{-1}$ exists must be wrong.

What is the condition that $(\hat{\Omega} - \omega\hat{I})$ has no inverse?

The inverse of an invertible matrix is given by

$$M^{-1} = \frac{(\text{cofactor } M)^T}{\det M}$$

As long as M is finite, so is its cofactor. So for the inverse M^{-1} to not exist, det M must be zero.

So for $(*)$ to have a solution $|V\rangle \neq |0\rangle$,

we must have

$$\boxed{\det(\hat{\Omega} - \omega\hat{I}) = 0}$$

This equation will determine the possible eigenvalues ω.

To find them, project (*) onto a basis:

$$\langle i | \hat{\Omega} - \omega \hat{I} | V \rangle = \langle i | 0 \rangle = 0$$

insert $\hat{I} = \sum_j |j\rangle\langle j|$

$$\Rightarrow \boxed{\sum_j (\Omega_{ij} - \omega \delta_{ij}) v_j = 0} \qquad (\#)$$

This is a coupled system of linear equations which we can solve for the components v_i of the eigenvectors once we found the eigenvalue ω.

The determinant of the matrix $\Omega_{ij} - \omega \delta_{ij}$ is an n^{th}-order polynomial in ω:

$$\det(\hat{\Omega} - \omega \hat{I}) = 0 \Leftrightarrow \boxed{\sum_{m=0}^{n} c_m \omega^m = 0}$$

The left hand side $P^{(n)}(\omega) = \sum_{m=0}^{n} c_m \omega^m$ is called the <u>characteristic polynomial</u> of $\hat{\Omega}$.

The polynomial looks different in different bases, but its roots, which are determined by the abstract equation (*), are <u>basis independent</u>.

Every n^{th} order polynomial has n complex roots $\omega_1, \omega_2, \ldots, \omega_n$. They need not be distinct, and in general they are not real. Once the eigenvalues are found, one solves the set of linear equations (#) to obtain the eigenvectors.