Example \( \hat{R} = R(\frac{\pi}{2}) \)

\[ \begin{align*}
\hat{R} |1\rangle &= |1\rangle \\
\hat{R} |2\rangle &= |3\rangle \\
\hat{R} |3\rangle &= |2\rangle
\end{align*} \]

\[ [R_{ij}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \]

Exercise: How can you describe the action of \( \hat{R} \) whose matrix elements in the same basis are given by \( [R_{ij}] = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \)?

Matrix forms of some specific operators

(1) Identity operator: \( \hat{I}_{ij} = \langle i|\hat{I}|j\rangle = \langle i|j \rangle = \delta_{ij} \equiv \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \)

(2) Projection operators:

Consider the expansion of an arbitrary ket \( |V\rangle \) as

\[ |V\rangle = \sum_{i=1}^{N} |i\rangle \langle i|V\rangle = \left( \sum_{i=1}^{N} |i\rangle \langle i| \right) |V\rangle \]

Since this is true for all \( |V\rangle \), the expression in brackets must be the identity operator:

\[ \hat{I} = \sum_{i=1}^{N} |i\rangle \langle i| = \sum_{i=1}^{N} \hat{P}_i \]

The object \( \hat{P}_i = |i\rangle \langle i| \) is called the projection operator for the ket \( |i\rangle \). Eq. (1) is called the completeness relation, it expresses the identity operator as a sum over projection operators. This will prove very valuable.
Now consider

\[ \hat{P}_i |V\rangle = |i\rangle \langle i| V\rangle = v_i |i\rangle \]

Clearly, \( \hat{P}_i \) is linear. Whatever \( |V\rangle \), \( \hat{P}_i (V) \) points in direction of \( |i\rangle \). \( \hat{P}_i \) projects out the component of \( |V\rangle \) along \( |i\rangle \). The completeness relation says that the sum of the projections of \( |V\rangle \) along the \( n \) basis directions reproduce the vector \( |V\rangle \).

\( \hat{P}_i \) can also act on bra:

\[ \langle V| \hat{P}_i = \langle V|i\rangle \langle i| = \langle i| V^* \]

We have

\[ \hat{P}_i \hat{P}_j = |i\rangle \langle i| j\rangle \langle j| = \delta_{ij} |i\rangle \langle j| = \delta_{ij} \hat{P}_j \]

Projectors along orthogonal directions give 0, projecting multiple times along the same direction reproduces the result of the first projection.

Example: polarization filters:

![Diagram of polarization filters](image-url)
The matrix element of $\hat{P}_i$ are

$$(\hat{P}_i)_{kl} = \langle k | i \rangle \langle i | l \rangle = \delta_{ki} \delta_{li}$$

$$= \begin{bmatrix}
0 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 0
\end{bmatrix}$$

$\Rightarrow$ only one non-zero matrix element of value 1 on the diagonal at position $i$.

Adding all projection operators fills the diagonal with 1's and thus reproduces the identity operator.

(3) Matrix elements of products of operators:

$$(\Omega \Lambda)_{ij} = \langle i | \hat{\Omega} \hat{\Lambda} | j \rangle = \langle i | \hat{\Omega} \hat{\Lambda} | j \rangle$$

$$= \langle i | \hat{\Omega} (\sum_{k=1}^{n} \langle k | \Lambda | k \rangle \hat{\Lambda}) | j \rangle$$

$$= \sum_{k=1}^{n} \langle i | \hat{\Omega} \hat{\Lambda} | k \rangle \langle k | \Lambda | j \rangle = \sum_{k=1}^{n} \Omega_{ik} \Lambda_{kj}$$

matrix multiplication!

(4) Adjoint of an operator:

For a ket $| \alpha V \rangle = | \alpha \rangle V$, the corresponding bra is

$$\langle \alpha V | = \langle V | \alpha^* \quad \text{(not } \langle V | a \rangle ! ! \text{)}$$
Similarly, for a ket \( |\hat{\Omega}v\rangle = |\hat{\Omega}v\rangle \), the corresponding bra is
\[
\langle \hat{\Omega}v | = \langle v | \hat{\Omega}^*
\]
which defines the adjoint operator \( \hat{\Omega}^* \).
As \( \hat{\Omega} \) turns \( |v\rangle \) in \( |v\rangle = \langle \hat{\Omega}v | \), \( \hat{\Omega}^* \) turns
\( \langle v | \) into \( \langle v | = \langle v | \hat{\Omega}^* \).
Its matrix elements are
\[
(\Omega^*)_ij = \langle i | \hat{\Omega}^* | j \rangle = \langle \Omega i | j \rangle = \langle j | \Omega^* i \rangle = \Omega^*_{ji}
\]
So the matrix for \( \hat{\Omega}^* \) is obtained from that for \( \hat{\Omega} \) by transposing columns and rows and complex conjugating each matrix element.

Adjoint of a product of operators:
\[
(\hat{\Omega} \hat{\Lambda})^* = \Lambda^* \hat{\Omega}^*
\]

(Proof: \( \langle v | (\hat{\Omega} \hat{\Lambda})^* \) = \( \langle \Lambda (\hat{\Omega}^*) v | = \langle \hat{\Omega}^* (\hat{\Lambda} v) | = \langle \hat{\Lambda} v | \Omega^* \)
\( = \langle v | \Lambda^* \hat{\Omega}^* = \langle v | \Lambda^* \hat{\Omega}^* \).

Example: consider \( \alpha_1 |v_1\rangle + \alpha_2 |v_2\rangle + \alpha_3 |v_3\rangle |v_4\rangle + \alpha_4 \hat{\Omega} |v_6\rangle \)
What is the adjoint?

Answer: \( \langle v_1 | \alpha_1^* = \langle v_1 | \alpha_2^* + \langle v_1 | v_4 \rangle \langle v_4 | v_5 \rangle + \alpha_4 \langle \hat{\Omega} | \hat{\Omega}^* \alpha_4 \)
\( = \langle v_2 | \alpha_2^* + \langle v_3 | \alpha_3^* + \langle v_4 | v_5 \rangle \hat{\Omega}^* + \hat{\Omega}^* | \Lambda^* \hat{\Omega}^* \alpha_4 \)

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**Definition:** A Hermitian operator is self-adjoint: $\hat{\Omega}^* = \hat{\Omega}$

**Definition:** An anti-Hermitian operator satisfies $\hat{\Omega}^* = -\hat{\Omega}$

**Definition:** A unitary operator satisfies $\hat{U}\hat{U}^* = \hat{I}$, i.e. $\hat{U}^* = \hat{U}^{-1}$

Since the inverse satisfies $\hat{U}^{-1}\hat{U} = \hat{U}\hat{U}^{-1} = \hat{I}$, we also have $\hat{U}^*\hat{U} = \hat{I}$. (This holds in finite dimensional vector spaces, but may not hold in infinite dimensional vector spaces without additional restrictions.)

- Any operator can be decomposed into Hermitian and anti-Hermitian parts:

$$\hat{\Omega} = \frac{\hat{\Omega} + \hat{\Omega}^*}{2} + \frac{\hat{\Omega} - \hat{\Omega}^*}{2}$$

  - Hermitian
  - Anti-Hermitian

- Any product of unitary operators is unitary.

$$(\hat{U}_1\hat{U}_2)^* = \hat{U}_2^* \hat{U}_1^* \Rightarrow (\hat{U}_1\hat{U}_2)^* (\hat{U}_1\hat{U}_2) = \hat{U}_2^* \hat{U}_1^* \hat{U}_1 \hat{U}_2$$

$$= \hat{U}_2^* \hat{U}_2 = \hat{I}$$

- Unitary operators preserve inner product:

$$|V_1\rangle = \hat{U} |V_1\rangle ; |V_2\rangle = \hat{U} |V_2\rangle \Rightarrow \langle V_2 | V_1\rangle = \langle \hat{U} V_2 | \hat{U} V_1\rangle$$

$$= \langle V_2 | \hat{U}^* \hat{U} V_1\rangle = \langle V_2 | V_1\rangle$$
Theorem: Both the columns of an \( n \times n \) unitary matrix and the rows of such a matrix form orthonormal basis sets of dimension \( n \).

Proof: \( \delta_{ij} = \langle i | \hat{U} | j \rangle = \langle i | \hat{U}^* \hat{U} | j \rangle \)

\[ = \sum_k \langle i | \hat{U}^* | k \rangle \langle k | \hat{U} | j \rangle \]

\[ = \sum_k (U^*)_i^k U^k_j = \sum_k U^*_{ki} U_{kj} \]

This proves the result for the columns.
(\( U_{kj} \) are the elements of the \( j \)th column.)

The proof for the rows follows similarly after substituting \( \hat{U}^* = \hat{U}^\dagger \).

I.7 Unitary transformations of operators

Under a unitary transformation \( |V\rangle \rightarrow \hat{U} |V\rangle \)
the matrix elements of an operator change as:

\[ \langle V' | \hat{S} | V \rangle \rightarrow \langle UV' | \hat{S} | UV \rangle = \langle V | \hat{U}^* \hat{S} \hat{U} | V \rangle \]

So instead of transforming the "state" \( |V\rangle \rightarrow \hat{U} |V\rangle \)
we can transform the operator \( \hat{S} \rightarrow \hat{U}^* \hat{S} \hat{U} \)

Since we leave the vectors alone and transform only the operators, this is called a passive transformation.
I.8. The eigenvalue problem

For each operator $\hat{S}$, there are certain kets that are simply rescaled (i.e., multiplied by a constant) when $\hat{S}$ acts on them:

$$\hat{S} |V\rangle = \omega |V\rangle \quad (*)$$

Any ket $|V\rangle$ with that property is called an eigenket of $\hat{S}$, and $\omega$ is called the eigenvalue of $\hat{S}$ for that ket. (*) is called an eigenvalue equation.

Example: Consider $\hat{S} = \hat{I}$

Since $\hat{I} |V\rangle = |V\rangle$

all vectors are eigenvalues of $\hat{I}$, and 1 is the only eigenvalue.

Example: Consider $\hat{S} = \hat{P}_V$ where $|V\rangle$ is normalized:

$$\hat{P}_V = |V\rangle\langle V|$$

1. Any ket $|\alpha V\rangle$ (parallel to $|V\rangle$) is an eigenket with eigenvalue 1:

$$\hat{P}_V |\alpha V\rangle = |\alpha V\rangle$$

2. Any ket $|V\perp\rangle$ perpendicular to $|V\rangle$ is an eigenket with eigenvalue 0:

$$\hat{P}_V |V\perp\rangle = 0$$

3. Any other ket (neither parallel nor perpendicular) is not an eigenket:

$$\hat{P}_V (\alpha |V\rangle + \beta |V\perp\rangle) = \alpha |V\rangle + \beta |V\perp\rangle$$
A systematic approach to finding all eigenvalues and eigenvectors of an operator:

The characteristic equation

Let's rewrite the eigenvalue equation as

\[(\hat{\Omega} - \omega \hat{1}) |\psi\rangle = |\psi\rangle \]  

If \((\hat{\Omega} - \omega \hat{1})^{-1}\) exists, we can operate with it on both sides, to get

\[|\psi\rangle = (\hat{\Omega} - \omega \hat{1})^{-1} |\psi\rangle \]

But this makes no sense: any finite operator (with finite matrix elements) maps the null vector onto itself. Hence the assumption that \((\hat{\Omega} - \omega \hat{1})^{-1}\) exists must be wrong.

What is the condition that \((\hat{\Omega} - \omega \hat{1})\) has no inverse?

The inverse of an invertible matrix is given by

\[M^{-1} = \left( \frac{\text{cofactor } M}{\det M} \right)^T \]

As long as \(M\) is finite, so is its cofactor. So for the inverse \(M^{-1}\) to not exist, \(\det M\) must be zero.

So for \((*)\) to have a solution \(|\psi\rangle \neq |\psi\rangle\),

we must have

\[\det (\hat{\Omega} - \omega \hat{1}) = 0\]
This equation will determine the possible eigenvalues \( \omega \).

To find them, project (\( \star \)) onto a basis:

\[
\langle i | \hat{S}^2 - \omega \hat{I} | V \rangle = \langle i | 0 \rangle = 0
\]

\[\uparrow \text{ insert } \hat{I} = \sum_j |j\rangle \langle j| \]

\[
\Rightarrow \sum_j (\Omega_{ij} - \omega \delta_{ij}) v_j = 0
\]

(\#)

This is a coupled system of linear equations which we can solve for the components \( v_i \) of the eigenvectors once we found the eigenvalue \( \omega \).

The determinant of the matrix \( \Omega_{ij} - \omega \delta_{ij} \) is an \( n \)th order polynomial in \( \omega \):

\[
\det (\hat{S}^2 - \omega \hat{I}) = 0 \iff \sum_{m=0}^{n} c_m \omega^m = 0
\]

The left-hand side \( \Phi^{(m)}(\omega) = \sum_{m=0}^{n} c_m \omega^m \) is called the characteristic polynomial of \( \hat{S}^2 \).

The polynomial looks different in different bases, but its roots, which are determined by this algebra equation (\( \star \)), are basis independent.

Every \( n \)th order polynomial has \( n \) complex roots \( \omega_1, \omega_2, \ldots, \omega_n \). They need not be distinct, and in general they are not real. Once the eigenvalue, are found, one solves the set of linear equations (\( \# \)) to obtain the eigenvectors.