Theorem The eigenvalues of a unitary operator lie on the unit circle in $\mathbb{C}$, i.e. are complex numbers with unit modulus; their eigenvectors are mutually orthogonal (for degenerate situations, this will be achieved in the same way as for Hermitian operators).

Proof (assuming no degeneracy for simplicity)

$$\hat{U}|u_i\rangle = u_i|u_i\rangle\ ,\quad \hat{U}|u_j\rangle = u_j|u_j\rangle$$

$$\langle u_j|\underbrace{U^+U}_{=\hat{I}}|u_i\rangle = u_i u_j^* \langle u_j|u_i\rangle$$

$$\Rightarrow (1-u_iu_j^*)\langle u_j|u_i\rangle = 0$$

For $i=j$: $1-u_iu_i^* = 0$ since $\langle u_i|u_i\rangle \neq 0 \Rightarrow |u_i|^2 = 1$

For $i\neq j$: Since $|u_i\rangle \neq |u_j\rangle \Rightarrow u_i \neq u_j \Rightarrow u_iu_j^* \neq u_iu_i^* = 1$
hence $u_iu_j^* \neq 1$ and therefore $\langle u_j|u_i\rangle = 0$ ◇

Diagonalization of Hermitian or unitary matrices:

Consider an orthonormal basis
$|1\rangle, |2\rangle \ldots |n\rangle$ in $\mathbb{V}^n(\mathbb{C})$

The orthonormalized eigenstates $|\omega_1\rangle, \ldots, |\omega_n\rangle$ of a Hermitian or unitary operator $\hat{\Omega}$ can be expanded in this basis as

$$|\omega_i\rangle = \sum_k u_{ik}|k\rangle$$

where u_{ik} is the k^{th} component of $|\omega_i\rangle$ in the basis $\{|k\rangle, k=1,\ldots n\}$.

In the basis $|\omega_i\rangle$ the operator is diagonal:

$$\langle \omega_i | \hat{\Omega} | \omega_j \rangle = \omega_i \delta_{ij} \quad \longleftrightarrow \quad \begin{pmatrix} \omega_1 & & & \\ & \omega_2 & & \\ & & \ddots & \\ & & & \omega_n \end{pmatrix}$$

$$\Big(= \sum_{k'} \langle k' | u^*_{ik'} | \hat{\Omega} | \Big(\sum_k u_{jk} | k \rangle \Big) \Big)$$

$$= \sum_{kk'} u^*_{ik'} \Omega_{k'k} u_{jk} = \left((\hat{U}^T)^+ \hat{\Omega}\, U^T \right)_{ij}$$

U^T is a unitary matrix made up from the orthonormal eigenvectors $|\omega_i\rangle$ in the basis $\{|k\rangle\}$. We see that $\hat{\Omega}$ is diagonalized by a unitary transformation with this unitary matrix.

$\Rightarrow$ Any Hermitian or unitary matrix Ω can be diagonalized by a unitary matrix U (built from the eigenvectors of Ω) by forming $U^+ \Omega U$.

$\Rightarrow$ <u>**Corollary:**</u> The determinant of a Hermitian or unitary operator is given by the product of its eigenvalues.

<u>**Proof:**</u> $\det(\hat{\Omega}) = \det(\hat{\Omega}\hat{I}) = \det(\hat{\Omega}\hat{U}\hat{U}^+)$

$= \det(\hat{U}^+ \hat{\Omega} U)$

Take for $\hat{U}$ the operator that diagonalizes $\hat{\Omega}$ and you are done.

Simultaneous diagonalization of two Hermitean operators

Theorem If $\hat{\Omega}$ and $\hat{\Lambda}$ are two commuting Hermitian operators, $[\hat{\Omega},\hat{\Lambda}]=0$, then there exists (at least) a basis of common eigenvectors that diagonalizes both.

Proof: • Consider first the case that at least one of the two operators, say $\hat{\Omega}$, is nondegenerate (i.e. each eigenvalue has only one normalized eigenvector):

$$\hat{\Omega}|\omega_i\rangle = \omega_i|\omega_i\rangle \quad \Rightarrow \quad \hat{\Lambda}\hat{\Omega}|\omega_i\rangle = \omega_i\hat{\Lambda}|\omega_i\rangle$$

since $[\hat{\Lambda},\hat{\Omega}]=0 \quad \Rightarrow \quad \hat{\Omega}(\hat{\Lambda}|\omega_i\rangle) = \omega_i(\hat{\Lambda}|\omega_i\rangle)$

$\Rightarrow$ $\hat{\Lambda}|\omega_i\rangle$ is also an eigenvector of $\hat{\Omega}$, with the same eigenvalue. Since the eigenvectors of $\hat{\Omega}$ were assumed to be unique, up to a scale $\Rightarrow$ $\hat{\Lambda}|\omega_i\rangle = \lambda_i|\omega_i\rangle$ ◇

• Now the more complex case where both operators have degenerate eigenvalues. Let us reorder the eigenstates such that in its eigenbasis $\hat{\Omega}$ is represented by the diagonal matrix

$$\hat{\Omega} \leftrightarrow \begin{pmatrix} \omega_1 & & & & & & \\ & \ddots & & & & & \\ & & \omega_1 & & & & \\ & & & \omega_2 & & & \\ & & & & \ddots & & \\ & & & & & \omega_k & \\ & & & & & & \ddots \\ & & & & & & & \omega_k \end{pmatrix}$$

Note that this eigenbasis is not unique: In the subspace $\mathbb{V}^{m_i}_{\omega_i} \equiv \mathbb{V}^{m_i}_{i}$ corresponding to eigenvalue ω_i, there exists an infinity of bases. Let us pick one such set and label the eigenstates as $|\omega_i,\alpha\rangle$ where $\alpha = 1,2,\ldots,m_i$.

In this basis, what are the matrix elements of $\hat{\Lambda}$?

Look at

$$\hat{\Omega}\hat{\Lambda}|\omega_i,\alpha\rangle \overset{\text{commute}}{=} \hat{\Lambda}\hat{\Omega}|\omega_i,\alpha\rangle = \omega_i\hat{\Lambda}|\omega_i,\alpha\rangle$$

So $\hat{\Lambda}|\omega_i,\alpha\rangle$ is an eigenstate of $\hat{\Omega}$ with eigenvalue ω_i, i.e. it lies in $\mathbb{V}_i^{m_i}$, but can be an arbitrary superposition of the basis states $\{|\omega_i,\alpha\rangle \,|\, \alpha=1,2,\dots m_i\}$. On the other hand, since eigenstates from different eigenvalues are orthogonal, $\langle\omega_j,\beta|\omega_i,\alpha\rangle = 0$ for $i\neq j$, for any α,β, we can conclude that $\langle\omega_j,\beta|\hat{\Lambda}|\omega_i,\alpha\rangle = 0$. Hence, in the chosen basis, $\hat{\Lambda}$ is represented by the block-diagonal matrix

$$\hat{\Lambda} \longleftrightarrow \begin{pmatrix} \underbrace{\boxed{\Lambda_1}}_{m_1}\Big\}m_1 & & & \\ & \boxed{\Lambda_2} & & \\ & & \ddots & \\ & & & \underbrace{\boxed{\Lambda_k}}_{m_k}\Big\}m_k \end{pmatrix}.$$

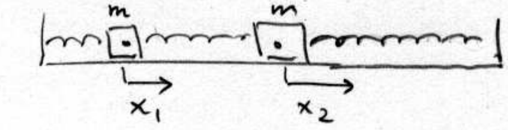
Now consider one of the block matrices Λ_i, acting on vectors in $\mathbb{V}_i^{m_i}$. As Λ_i acts on some $\hat{\Omega}$-eigenvector $|\omega_i,\alpha\rangle$, it turns it into some other vector in $\mathbb{V}_i^{m_i}$. Λ_i is Hermitian, since $\hat{\Lambda}$ is. So we can diagonalize it by trading the basis $\{|\omega_i,\alpha\rangle \,|\, \alpha=1,2,\dots m_i\}$ for an eigenbasis of Λ_i. Since the vectors of their eigenbasis are linear superpositions of the $\{|\omega_i,\alpha\rangle \,|\, \alpha=1,2,\dots,m_i\}$, all of which are $\hat{\Omega}$-eigenstates with eigenvalue ω_i; the Λ_i-eigenstates remain eigenstates of $\hat{\Omega}$ with eigenvalue ω_i. In the eigenbasis of $\hat{\Lambda}$, made from the eigenbasis set of Λ_i, $\hat{\Omega}$ and $\hat{\Lambda}$ are now both represented by diagonal matrices:

$$\hat{\Omega} \leftrightarrow \begin{pmatrix} \underbrace{\omega_1 \cdots \omega_1}_{m_1} & & & \\ & \underbrace{\omega_2 \cdots}_{m_2} & & \\ & & \ddots & \\ & & & \underbrace{\omega_k \cdots \omega_k}_{m_k} \end{pmatrix} \qquad \hat{\Lambda} \leftrightarrow \begin{pmatrix} \underbrace{\lambda_1^{(1)} \cdots \lambda_1^{(m_1)}}_{m_1} & & & \\ & \lambda_2^{(1)} \cdots & & \\ & & \ddots & \\ & & & \underbrace{\lambda_k^{(1)} \cdots \lambda_k^{(m_k)}}_{m_k} \end{pmatrix}$$

If inside each $\mathbb{V}_i^{m_i}$, $\hat{\Lambda}$ is non-degenerate, then this eigenbasis is unique. The eigenstates are the uniquely labelled by the eigenvalue ω of $\hat{\Omega}$ and λ of $\hat{\Lambda}$: $|\omega,\lambda\rangle$. If Λ_i is still degenerate, then we need one or more additional labels: $|\omega,\lambda,\gamma,\ldots\rangle$, corresponding to eigenvalues of additional operators $\hat{\Gamma},\ldots$, that commute with <u>both</u> $\hat{\Omega}$ and $\hat{\Lambda}$. We will henceforth assume that a <u>complete set of commuting operators</u>, whose eigenvalue identify the eigen-basis vectors uniquely, always exists, even if n is infinite. (For finite n, this is always true.)

Example from classical mechanics:

Two equal masses coupled to each other and two walls by springs:



x_1, x_2 measure the displacement of the masses from their equilibrium points.
All springs have the same spring constant k

Equations of motion

$$\ddot{x}_1 = -\frac{2k}{m} x_1 + \frac{k}{m} x_2 \qquad (1)$$
$$\ddot{x}_2 = +\frac{k}{m} x_1 - \frac{2k}{m} x_2 \qquad (2)$$

Find $x_1(t), x_2(t)$, given initial positions and velocities.
Assume zero initial velocities for simplicity. So initial conditions are $x_1(0), x_2(0)$.

Reformulate in language of linear vector spaces:

$$\begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} = \begin{pmatrix} \Omega_{11} & \Omega_{12} \\ \Omega_{21} & \Omega_{22} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad \text{or} \quad |\ddot{\vec{x}}(t)\rangle = \hat{\Omega} |\vec{x}(t)\rangle \quad (*)$$

$$\Omega_{11} = \Omega_{22} = -\frac{2k}{m} \qquad \Omega_{12} = \Omega_{21} = k/m$$

$|\vec{x}\rangle \in \mathbb{V}^2(\mathbb{R})$ (2 real components) $\rightarrow$ 2-dimensional basis

$\hat{\Omega}$ is Hermitian

Let's introduce the following basis:

$|1\rangle \leftrightarrow \begin{pmatrix} 1 \\ 0 \end{pmatrix} \leftrightarrow$ [first mass displaced by one length unit, second mass undisplaced]

$|2\rangle \leftrightarrow \begin{pmatrix} 0 \\ 1 \end{pmatrix} \leftrightarrow$ [first mass undisplaced, second mass displaced by unity]

Let us project $(*)$ onto $|1\rangle$:

$$\langle 1|\ddot{\vec{x}}(t)\rangle = \langle 1|\hat{\Omega}|\vec{x}(t)\rangle \Leftrightarrow (1,0)\begin{pmatrix}\ddot{x}_1\\ \ddot{x}_2\end{pmatrix} = (1,0)\begin{pmatrix}-\frac{2k}{m} & \frac{k}{m}\\ k/m & -\frac{2k}{m}\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix}$$

$$= (1,0)\begin{pmatrix}-\frac{2k}{m}x_1 + \frac{k}{m}x_2\\ +\frac{k}{m}x_1 - \frac{2k}{m}x_2\end{pmatrix}$$

$$\Leftrightarrow \underbrace{\ddot{x}_1 = -\frac{2k}{m}x_1 + \frac{k}{m}x_2} \quad \text{first E.O.M. } (\underline{1})$$

Similarly, the second E.O.M. is obtained by projecting $(*)$ onto $|2\rangle$.

An arbitrary state of the system is given by

$$\begin{pmatrix}x_1(t)\\ x_2(t)\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix}x_1(t) + \begin{pmatrix}0\\1\end{pmatrix}x_2(t)$$

or $\boxed{|\vec{x}(t)\rangle = x_1(t)|1\rangle + x_2(t)|2\rangle}$

In this basis, the matrix $\begin{pmatrix}-2k/m & k/m\\ k/m & -2k/m\end{pmatrix}$ represents the operator $\hat{\Omega}$ in $(*)$.

The basis $|1\rangle, |2\rangle$ is desirable for giving the final solution in an explicit form. However, solution of the initial value problem is not easy in this basis, since $\hat{\Omega}$ is non-diagonal. The solution is obtained much more easily in a basis of eigenstates of $\hat{\Omega}$.

There will be two eigenvectors; let us call them $|\mathrm{I}\rangle$ and $|\mathrm{II}\rangle$:

$$\hat{\Omega}|\mathrm{I}\rangle = -\omega_{\mathrm{I}}^2|\mathrm{I}\rangle \quad ; \quad \hat{\Omega}|\mathrm{II}\rangle = -\omega_{\mathrm{II}}^2|\mathrm{II}\rangle$$

(We call the eigenvalues $-\omega_{\mathrm{I,II}}^2$ because we know from kindergarten that this will be convenient; at this point, nothing tells us that $\omega_{\mathrm{I,II}}$ will not be imaginary.

We only know from Hermiticity of $\hat{\Omega}$ that ω_I^2, ω_{II}^2 are real, but not that they are positive.)

Let's solve the eigenvalue problem to find the eigenvalues:

The characteristic equation $\det(\hat{\Omega} + \omega^2 \hat{I}) = 0$

becomes $\begin{vmatrix} -\frac{2k}{m} + \omega^2 & k/m \\ k/m & -\frac{2k}{m} + \omega^2 \end{vmatrix} = 0 \Rightarrow \left(-\frac{2k}{m} + \omega^2\right)^2 - \frac{k^2}{m^2} = 0$

$$\Rightarrow \omega^2 - \frac{2k}{m} = \pm \frac{k}{m}$$

$$\Rightarrow \boxed{\omega_I^2 = \frac{k}{m}, \quad \omega_{II}^2 = \frac{3k}{m}}$$

So the eigenfrequencies are $\omega_I = \sqrt{\frac{k}{m}}$, $\omega_{II} = \sqrt{3}\,\omega_I$.

The corresponding eigenvectors are:

$\underset{\sim}{\omega_I}$: $\begin{pmatrix} -2\omega_I^2 & \omega_I^2 \\ \omega_I^2 & -2\omega_I^2 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = -\omega_I^2 \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$

$$\Rightarrow \begin{aligned} -2\omega_I^2 a_1 + \omega_I^2 a_2 &= -\omega_I^2 a_1 && \Rightarrow a_2 = a_1 \\ \omega_I^2 a_1 - 2\omega_I^2 a_2 &= -\omega_I^2 a_2 && \Rightarrow a_1 = a_2 \text{ (redundant)} \end{aligned}$$

$$\Rightarrow \boxed{|I\rangle \leftrightarrow \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}} \text{ (normalized)}$$

$\underset{\sim}{\omega_{II}}$: $\begin{pmatrix} -\frac{2}{3}\omega_{II}^2 & \frac{1}{3}\omega_{II}^2 \\ \frac{1}{3}\omega_{II}^2 & -\frac{2}{3}\omega_{II}^2 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = -\omega_{II}^2 \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$

$$\Rightarrow \left.\begin{aligned} -\tfrac{2}{3}a_1 + \tfrac{1}{3}a_2 &= -a_1 \\ \tfrac{1}{3}a_1 - \tfrac{2}{3}a_2 &= -a_2 \end{aligned}\right\} \Rightarrow \begin{aligned} &\tfrac{1}{3}a_2 = -\tfrac{1}{3}a_1 \Rightarrow a_1 = -a_2 \\ &\tfrac{1}{3}a_1 = -\tfrac{1}{3}a_2 \Rightarrow \text{redundant} \end{aligned}$$

$$\Rightarrow \boxed{|II\rangle \leftrightarrow \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix}}$$

Expanding now the system's state vector in this new basis as

$$|\vec{x}(t)\rangle = x_I(t)|I\rangle + x_{II}(t)|II\rangle$$

the components x_I and x_{II} will evolve as follows:

$$|0\rangle = \left(\frac{d^2}{dt^2} - \hat{\Omega}\right)|\vec{x}(t)\rangle = \left(\frac{d^2}{dt^2} - \hat{\Omega}\right)\Big[x_I(t)|I\rangle + x_{II}(t)|II\rangle\Big]$$

$$= (\ddot{x}_I - x_I\hat{\Omega})|I\rangle + (\ddot{x}_{II} - x_{II}\hat{\Omega})|II\rangle$$

$$= (\ddot{x}_I + \omega_I^2 x_I)|I\rangle + (\ddot{x}_{II} + \omega_{II}^2 x_{II})|II\rangle$$

Projecting with $\langle I|$ gives $\ddot{x}_I = -\omega_I^2 x_I$

" " $\langle II|$ gives $\ddot{x}_{II} = -\omega_{II}^2 x_{II}$

These equations are decoupled and easily solved. With vanishing initial velocity the solutions are

$$\boxed{x_i(t) = x_i(0)\cos(\omega_i t) \qquad i = I, II}$$

$$\Rightarrow |\vec{x}(t)\rangle = \underbrace{x_I(0)}_{\langle I|\vec{x}(0)\rangle}\cos(\omega_I t)|I\rangle + \underbrace{x_{II}(0)}_{\langle II|\vec{x}(0)\rangle}\cos(\omega_{II} t)|II\rangle$$

$$= |I\rangle\langle I|\vec{x}(0)\rangle\cos(\omega_I t) + |II\rangle\langle II|\vec{x}(0)\rangle\cos(\omega_{II} t)$$

From this we can get $x_1(t)$ and $x_2(t)$ by projecting with $\langle 1|$ and $\langle 2|$, respectively:

$$x_1(t) = \langle 1|\vec{x}(t)\rangle = \langle 1|I\rangle\langle I|\vec{x}(0)\rangle\cos(\omega_I t) + \langle 1|II\rangle\langle II|\vec{x}(0)\rangle\cos(\omega_{II} t)$$
$$= \frac{1}{\sqrt{2}}\left[x_I(0)\cos(\omega_I t) + x_{II}(0)\cos(\omega_{II} t)\right]$$

$$x_2(t) = \langle 2|\vec{x}(t)\rangle = \langle 2|I\rangle\langle I|\vec{x}(0)\rangle\cos(\omega_I t) + \langle 2|II\rangle\langle II|\vec{x}(0)\rangle\cos(\omega_{II} t)$$
$$= \frac{1}{\sqrt{2}}\left[x_I(0)\cos(\omega_I t) - x_{II}(0)\cos(\omega_{II} t)\right]$$

To complete we only need to express $x_I(0)$ and $x_{II}(0)$ in terms of $x_1(0)$ and $x_2(0)$:

$$x_I(0) = \langle I|\vec{x}(0)\rangle = \left(\tfrac{1}{\sqrt{2}}, \tfrac{1}{\sqrt{2}}\right)\begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix} = \frac{x_1(0)+x_2(0)}{\sqrt{2}}$$

$$x_{II}(0) = \langle II|\vec{x}(0)\rangle = \left(\tfrac{1}{\sqrt{2}}, -\tfrac{1}{\sqrt{2}}\right)\begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix} = \frac{x_1(0)-x_2(0)}{\sqrt{2}}$$

$$\Rightarrow \quad x_1(t) = \frac{x_1(0)+x_2(0)}{2}\cos(\omega_I t) + \frac{x_1(0)-x_2(0)}{2}\cos(\omega_{II} t)$$

$$x_2(t) = \frac{x_1(0)+x_2(0)}{2}\cos(\omega_I t) - \frac{x_1(0)-x_2(0)}{2}\cos(\omega_{II} t)$$

Or in matrix form:

$$\boxed{\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} \frac{1}{2}\left(\cos\left(\sqrt{\frac{k}{m}}t\right)+\cos\left(\sqrt{\frac{3k}{m}}t\right)\right) & \frac{1}{2}\left(\cos\left(\sqrt{\frac{k}{m}}t\right)-\cos\left(\sqrt{\frac{3k}{m}}t\right)\right) \\ \frac{1}{2}\left(\cos\left(\sqrt{\frac{k}{m}}t\right)-\cos\left(\sqrt{\frac{3k}{m}}t\right)\right) & \frac{1}{2}\left(\cos\left(\sqrt{\frac{k}{m}}t\right)+\cos\left(\sqrt{\frac{3k}{m}}t\right)\right) \end{pmatrix}\begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix}}$$

Here is a summary of the steps:

(1) Solve the eigenvalue problem for $\hat{\Omega}$

(2) Find $x_I(0) = \langle I|\vec{x}(0)\rangle$ and $x_{II}(0) = \langle II|x(0)\rangle$ from the initial conditions.
The initial state is $|\vec{x}(0)\rangle = |I\rangle x_I(0) + |II\rangle x_{II}(0)$ (*)

(3) To find t-dependence $|\vec{x}(t)\rangle$, simply take the r.h.s. of (*) and multiply each coefficient $x_i(0)$ ($i = I, II$) by $\cos(\omega_i t)$

(4) To find the positions $x_1(t)$ and $x_2(t)$ of the masses, project the solution on $\langle 1|$ and $\langle 2|$ ($\langle 1|\vec{x}(t)\rangle = x_1(t)$, $\langle 2|\vec{x}(t)\rangle = x_2(t)$)

The propagator

The solution of this problem has a remarkable structure:

- The final state vector is obtained from the initial state vector by multiplication with a matrix, and this matrix is independent of the initial state.

This matrix is called the propagator.

We can view our solution as the representation in the $|1\rangle, |2\rangle$ basis of the abstract relation

$$\boxed{|\vec{x}(t)\rangle = \hat{U}(t)\,|x(0)\rangle}$$

We read off from the underlined equation of p. 41:

$$\hat{U}(t) = |I\rangle\langle I|\cos(\omega_I t) + |II\rangle\langle II|\cos(\omega_{II} t)$$

$$= \sum_{i=I}^{II} |i\rangle\langle i|\cos(\omega_i t) \qquad (✡)$$

This is a sum over projectors on the eigenvectors with weight $\cos(\omega_i t)$. This structure is very generic and embodies the key features of time evolution also in quantum mechanics.

The matrix elements of $\hat{U}(t)$ in the $|1\rangle, |2\rangle$ basis are the matrix entries in the framed equ. on p. 42.

So to solve the problem $|\ddot{x}\rangle = \hat{\Omega}|x\rangle$

(1) solve the eigenvalue problem of $\hat{\Omega}$

(2) construct the propagator $\hat{U}(t)$ from eigenvectors and eigenvalues as in (✡)

(3) write down $|x(t)\rangle = \hat{U}(t)\,|x(0)\rangle$. That's it!

The eigenkets $|I\rangle, |II\rangle$ are abstract vectors describing what in classical mechanics are called "normal modes". They evolve "trivially":

$$|I(t)\rangle = \cos(\omega_I t)\,|I\rangle$$
$$|II(t)\rangle = \cos(\omega_{II} t)\,|II\rangle$$

In the eigenbasis (normal mode representation) the propagator is diagonal:

$$\hat{U}(t) \underset{\substack{I,II\\ \text{basis}}}{\longleftrightarrow} \begin{pmatrix} \cos(\omega_I t) & 0 \\ 0 & \cos(\omega_{II} t) \end{pmatrix}$$

The central problem in Quantum Mechanics is to solve the Schrödinger equation:

$$i\hbar|\dot{\psi}\rangle = \hat{H}\,|\psi\rangle \qquad \left(\hbar = \frac{h}{2\pi}\right)$$

$\hat{H}$ is called the Hamiltonian, $|\psi\rangle$ the state vector of the system. (All the conceptual complexities of QM have to do with how one computes physical observables from that state vector.)

The problem is to find $|\psi(t)\rangle$ given $|\psi(0)\rangle$.

This problem is formally <u>very</u> similar to the classical 2-oscillator problem we just solved:

(1) Solve the eigenvalue problem of $\hat{H}$

(2) Find the propagator $\hat{U}(t)$ in terms of eigenvectors and eigenvalues of $\hat{H}$.

(3) $|\psi(t)\rangle = \hat{U}(t)\,|\psi(0)\rangle$. That's it!