Example: A normal mode problem in Hilbert space

Consider the displacement $f(x,t)$ of a string of length L clamped at its ends $x=0$ and $x=L$: $f(0,t)=f(L,t)=0 \ \forall t$.

It obeys the classical E.O.M. (we measure t in length units, $vt \mapsto t$ (v = wave propagation speed))

$$\boxed{\frac{\partial^2 f(x,t)}{\partial t^2} - \frac{\partial^2 f(x,t)}{\partial x^2} = 0} \qquad (*)$$

We assume that at $t=0$ the displacement is $f(x,0)$ and the velocity $\dot{f}(x,0) \equiv \frac{\partial f}{\partial t}(x,0) = 0$.

Solve for $f(x,t)$!

We identify $f(x,t)$ as the $|x\rangle$-basis components of a time-dependent vector $|f(t)\rangle$ in Hilbert space. Then $-\frac{\partial^2}{\partial x^2} f(x,t)$ are the x-space components of the vector $\hat{K}^2 |f(t)\rangle$, and $\frac{\partial^2}{\partial t^2} f(x,t)$ are the x-space components of $\frac{d^2}{dt^2}|f(t)\rangle$.

The wave eqn. $(*)$ thus becomes the x-representation of the time-dependent evolution equation

$$\boxed{\frac{d^2}{dt^2}|f(t)\rangle + \hat{K}^2 |f(t)\rangle = 0}. \qquad (**)$$

We solve it with our 3-step weight-loss program:

(1) Solve the eigenvalue problem of $\hat{K}^2$

(2) Construct the propagator $\hat{U}(t)$ in terms of eigenvalues and eigenvectors

(3) $|f(t)\rangle = \hat{U}(t)|f(0)\rangle$.

The eigenvalue equation is

$$\hat{K}^2|f\rangle = k^2|f\rangle$$

Its x-space matrix elements are

$$-\frac{d^2}{dx^2} f_k(x) = k^2 f_k(x)$$

The solutions are $\quad f_k(x) = A\cos(kx) + B\sin(kx)$

(we prefer this over $A' e^{ikx} + B' e^{-ikx}$ since it allows easier implementation of $f(0) = f(L) = 0.$)

Since $f_k(0) = 0 \Rightarrow A = 0$

Since $f_k(L) = 0 \Rightarrow B\sin(kL) = 0 \Rightarrow \boxed{kL = m\pi\,,\ m = 1, 2, 3, \ldots}$

$\Rightarrow$ <u>discrete eigenvalue spectrum!</u> (otherwise $B = 0$ and $f_k \equiv 0$).

Negative m values are redundant: $\sin(-x) = -\sin(x)$ correspond to the same vector $|f\rangle$.

Normalized solutions:

$$|m\rangle \leftrightarrow \quad f_m(x) = \sqrt{\frac{2}{L}}\sin\left(m\pi\frac{x}{L}\right) \qquad (\text{real!}) \Rightarrow \boxed{k_m = \frac{m\pi}{L}}$$

Eigenvalues!

$$\langle f_m \underset{\substack{\uparrow \\ \hat{I}_x}}{|} f_{m'}\rangle = \int_0^L dx\, f_m(x)\, f_{m'}(x) = \delta_{mm'}$$

The $|m\rangle$ basis diagonalizes $\hat{K}^2$. In the $|m\rangle$-basis Eq. (**) reads

$$\frac{d^2}{dt^2}\langle m|f(t)\rangle = -\left(\frac{m\pi}{L}\right)^2\langle m|f(t)\rangle \,, \quad m = 1, 2, \ldots$$

With vanishing initial velocities the solution is

$$\boxed{\langle m|f(t)\rangle = \langle m|f(0)\rangle \cos\left(\frac{m\pi}{L}t\right)}$$

$$\Rightarrow \boxed{|f(t)\rangle = \sum_{m=1}^{\infty} |m\rangle\langle m|f(t)\rangle = \sum_{m=1}^{\infty} |m\rangle\langle m|f(0)\rangle \cos(\omega_m t) \quad \left(\omega_m = \frac{m\pi}{L}\right)}$$

or
$$\boxed{\hat{\mathcal{U}}(t) = \sum_{m=1}^{\infty} |m\rangle\langle m| \cos(\omega_m t)}$$

The solution $|f(t)\rangle = \hat{\mathcal{U}}(t)|f(0)\rangle$ becomes in the x-basis

$$\langle x|f(t)\rangle = f(x,t) = \langle x|\hat{\mathcal{U}}(t)|f(0)\rangle = \int_0^L dx' \langle x|\hat{\mathcal{U}}(t)|x'\rangle f(x';0)$$

Now

$$\langle x|\hat{\mathcal{U}}(t)|x'\rangle = \sum_{m=1}^{\infty} \langle x|m\rangle\langle m|x'\rangle \cos(\omega_m t)$$

$$= \frac{2}{L}\sum_{m=1}^{\infty} \sin\left(\frac{m\pi}{L}x\right) \sin\left(\frac{m\pi}{L}x'\right) \cos(\omega_m t)$$

$$\Rightarrow \boxed{f(x,t) = \frac{2}{L}\sum_{m=1}^{\infty} \sin(k_m x)\cos(\omega_m t) \underbrace{\int_0^L dx' \sin(k_m x') f(x';0)}_{\text{Fourier components } f_{k_m}(0)} \quad \left(k_m = \frac{m\pi}{L}\right)}$$

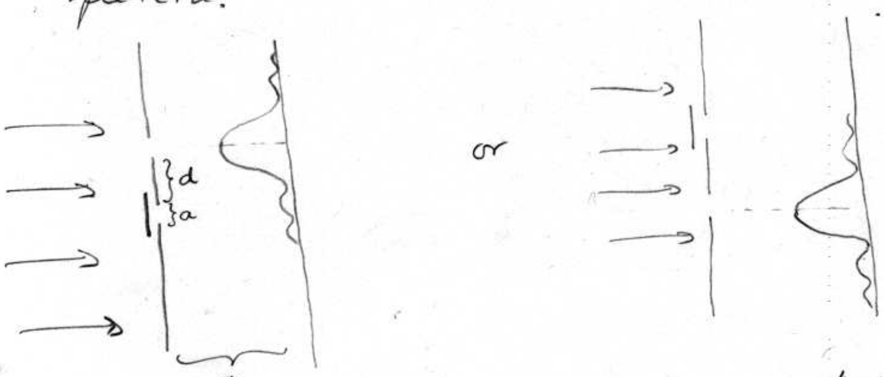
The m^{th} normal mode oscillates with wavenumber $k_m = \frac{m\pi}{L}$ in space (along the length of the string) and with frequency ω_m in time.

Chapter III: What's wrong with classical physics? *)

The 2-slit experiment:

Consider a wall with 2 circular holes of diameter a and separated by distance d, illuminated from one side by a monochromatic plane wave of light (laser). On the other side, at some distance $D \ggg \lambda$ and parallel to this wall, we have a finely pixellated plane of CCDs that detect the intensity of light falling on any point on this plane.

From classical optics we know that when we close one or the other of the two holes, we see a simple diffraction pattern:



(1)

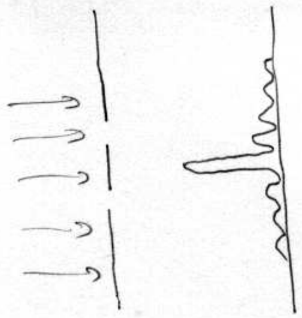
$$I = \langle |\vec{A}_1|^2 \rangle_t \qquad\qquad I = \langle |A_2|^2 \rangle_t$$

$$\vec{A}_1(\vec{r}, t) = A_0 \int_{\text{hole }1} d^2 r_1 \, e^{i(\vec{k}\cdot(\vec{r}-\vec{r}_1)-\omega t)} \qquad \vec{A}_2(\vec{r}, t) = A_0 \int_{\text{hole }2} d^2 r_2 \, e^{i(\vec{k}\cdot(\vec{r}-\vec{r}_2)-\omega t)}$$

and when we open both slits. we observe interference:

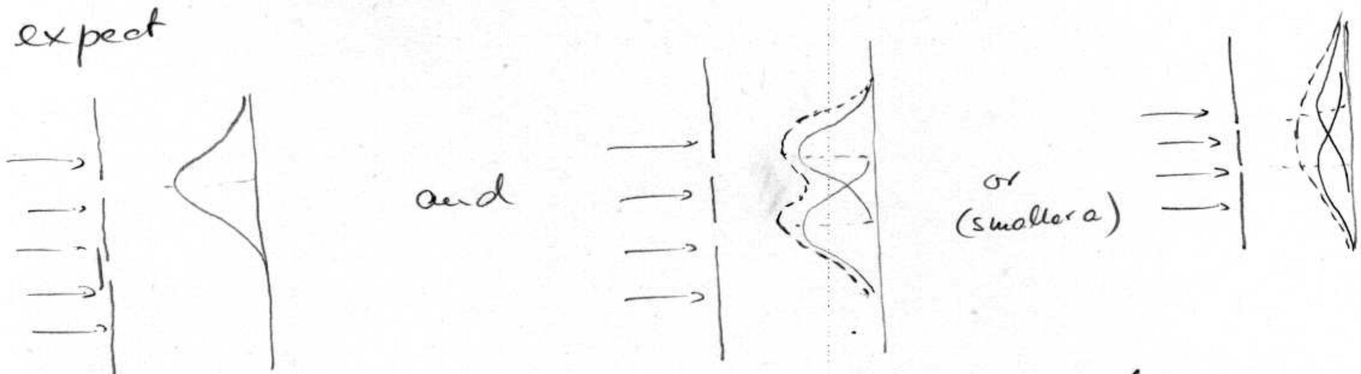
$$I = |\vec{A}_1 + \vec{A}_2|^2 \neq |A_1|^2 + |A_2|^2$$

*) Inspired by a series of essays by David Mermin in Physics Today.



(2)

This clearly is in contradiction with a picture where the incoming light is considered to be a homogeneous stream of tiny billiard balls ("photons"). In that case we would expect



(3)

i.e. both the diffraction and interference patterns disappear.

So light is a wave, right?

Well, let's turn down the intensity of our laser, — way down, until the energy flowing through each slit per second is of order $\hbar\omega = \hbar c k = \frac{hc}{\lambda}$, and look at the observed distribution of energy deposited on the screen after 1, 2, 3, ... 60 seconds. Will the measured pattern look like those shown in (1), (2), (3)?

(Do the computer simulation "Interference" from Thomas Moore's "Six Ideas that Shaped Physics, Unit Q)

The answer is: No! What we see instead is an almost random-looking distribution of pointlike hits (single pixels of the screen registering arriving energy), just as expected from the billiard-ball picture of a stream of photons! So light is a gas of photons, right?

Clearly we are drawing contradictory conclusions from our observations. What the hell is going on? Well, let's keep the intensity low and observe a little longer (hours, days, weeks ...). After waiting long enough, we see the diffraction and interference patterns emerge – initially with large statistical fluctuations, but eventually very smooth, just as predicted by classical wave optics.

So what do we conclude now?

(i) The deposition of energy on the detection screen is a stochastic and highly localized ("quantized") phenomenon.

(ii) The probability distribution controlling this stochastic process is smooth, delocalized and shows wavelike interference patterns.

(iii) The intensity, averaged over sufficiently large times such that very large numbers of energy quanta are registered, measures this probability distribution. It is calculated as $I \sim \langle |\vec{E}|^2 \rangle$ as the square of an amplitude $\sim \vec{E}$, describing the electric field of the electromagnetic wave; this <u>amplitude</u> is additive for multiple light sources (<u>"superposition principle"</u>). Interference is caused by the $\vec{r}$-t-dependence of the complex phase of this amplitude

This interpretation remains correct when you replace the laser beam by a monoenergetic electron beam in the double slit experiment. Again the electron hits on the screen are localized and described by a stochastic process, whose probability distribution shows diffraction and interference phenomena. The only difference is a (much) smaller wavelength. So electrons in the double slit experiment show wavelike behavior, just as a beam of photons ("light") does. The wavelike behavior becomes visible only when a large number of photons/electrons is measured. It is not described by classical mechanics.

It gets weirder: The interference phenomenon only happens when both slits are open, but clearly a photon/electron can only pass through one hole at the time. One can design "proximity detectors" that determine through which hole the photon/electron passes, without significantly affecting the particle's trajectory. When one does this (i.e. determines which hole the particle went through) the interference pattern disappears (show simulation). It only appears when we have both holes open and <u>don't monitor the holes</u>. In that case each photon/electron, even though it surely must pass through only one of the two holes, "knows" about the existence of an alternate path, and this existence of an alternate path (or the lack of information about which path was taken) modifies the probability distribution for its impact distribution on the screen such that it develops interference fringes (this implies, e.g., reduced probabilities to hit near the interference minima compared to the situation where we know through which hole the particle went!). So knowing less about the electron's path increases the probability in some places and decreases its probability in other places compared to when we know through which hole it went.

This is a manifestation of what some people call "quantum weirdness".

Quantum mechanics is the theory which correctly and accurately predicts these probability distributions (including all interference effect) through a complex wave amplitude ("wave function") whose square gives the probability. It is one of the most successful physical theories.