Physics 7501: Homework Set No. 12

Due date: Tuesday, December 8, 2015, 5:00pm in PRB 3018 (Fuyan Lu's office)

Total point value of set: 100 points

This is a practice exam, taken from the QM I course I gave in 2012. Try to finish it in 2 hours, then polish it and hand it in as a homework.

Problem 1 (20 points):

The Pauli spin operators $\hat{\sigma}_i$ ($i = x, y, z$ or $i = 1, 2, 3$) satisfy the commutation algebra

$$[\hat{\sigma}_k, \hat{\sigma}_l]_- = 2i \sum_{m=1}^{3} \epsilon_{klm} \hat{\sigma}_m, \qquad [\hat{\sigma}_k, \hat{\sigma}_l]_+ = 2\delta_{kl} \hat{\mathbf{1}}.$$

(i) (6 pts.) Using the matrix representation

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

of these operators, show that in all normalized states $|\psi\rangle$ the expectation value $\vec{\boldsymbol{s}} \equiv \langle\psi|\hat{\vec{\boldsymbol{\sigma}}}|\psi\rangle$ satisfies the constraint $\vec{\boldsymbol{s}}^2 = 1$.

(ii) (7 pts.) Establish upper and lower limits for the uncertainty $\Delta\sigma_i$ of any of the three spin observables ($i = 1, 2, 3$) in an arbitrary normalized state $|\psi\rangle$.

(iii) (7 pts.) Work out the uncertainty relation between $\hat{\sigma}_x$ and $\hat{\sigma}_y$. What is the minimal value of the uncertainty product of the two observables?
(*Hint:* This part of the problem is solved most efficiently by doing it algebraically, without using the matrix representation of the Pauli operators.)

Problem 2 (10 points):

The hydrogen atom is described by the Hamiltonian

$$\hat{H}(\hat{\vec{\boldsymbol{R}}}_p, \hat{\vec{\boldsymbol{P}}}_p; \hat{\vec{\boldsymbol{R}}}_e, \hat{\vec{\boldsymbol{P}}}_e) = \frac{\hat{\vec{\boldsymbol{P}}}_p^2}{2m_p} + \frac{\hat{\vec{\boldsymbol{P}}}_e^2}{2m_e} - \frac{e^2}{|\hat{\vec{\boldsymbol{R}}}_p - \hat{\vec{\boldsymbol{R}}}_e|}.$$

(a) (5 pts.) Write down the expansion of a general state $|\psi\rangle$ describing this proton+electron 2-particle system in terms of a basis of 2-particle states constructed from the 1-particle eigenstates of the position operators $\hat{\vec{\boldsymbol{R}}}_p = (\hat{X}_p, \hat{Y}_p, \hat{Z}_p)$ and $\hat{\vec{\boldsymbol{R}}}_e = (\hat{X}_e, \hat{Y}_e, \hat{Z}_e)$. Do you need to (anti-)symmetrize these basis states? (Explain!) Write down the eigenvalue equations that define your basis states.

(b) (5 pts.) Write down completeness and orthogonality relations for the 2-particle basis states defined in part (a).

Problem 3 (60 points):

Two ^{6}Li atoms are trapped in neighboring maxima of $|\vec{\boldsymbol{E}}|^2$ of an optical trap. The optical trap consists of a standing wave pattern (with electric field vector $\vec{\boldsymbol{E}}(\vec{\boldsymbol{x}}, t)$) made from laser light with wavelength 1064 nm (Nd:YAG laser), with intensity maxima that form a regular cubic lattice. The trapping force acting on the atoms can be approximated by a harmonic oscillator potential whose natural frequency depends on multiple parameters (mass and polarizability of the atoms, wavelength and intensity of the light) and is hard to calculate. Let us assume for this problem an effective spring constant of 0.1 pN/nm (according to Wikipedia a typical value for a 1 W laser). We can assume that both ^{6}Li atoms occupy the ground states of their respective 3-d harmonic oscillator potentials.

(i) (5 pts.) Does the wavefunction describing the two ^{6}Li atoms need to be symmetrized, antisymmetrized, or neither? Answer the question in principle, leaving practical considerations aside (we'll get to them in part **(iii)** below), and give a conclusive argument (not a 10-page derivation!) that justifies your answer.

(ii) (15 pts.) Write down the 3-dimensional harmonic wavefunctions for each of the two atoms as well as the 2-atom wavefunction that describes our system. Assume that the minima of the potentials trapping the atoms both lie on the x-axis.

(iii) (5 pts.) We learned in class that in practice (anti-)symmetrization can usually be ignored if the two particles are very well separated. Define a unitless parameter δ that allows you to assess whether the two trapped atoms are "well separated" (define it such that good separation implies $\delta \gg 1$), and work out its numerical value for the case at hand. Are the two atoms "well-separated"?

(iv) (20 pts.) Suppose you want to compute the probability density for finding one of the two atoms near the position of one of the two intensity maxima that trap the atoms. Compute the fractional error that arises from ignoring (anti-)symmetrization of the 2-particle state. Interpret your result.

(v) (15 pts.) Repeat for the probability density for finding one of the two atoms at the electric field node between the two intensity maxima. How large is the percentage error from ignoring (anti-)symmetrization in this case? Explain!

Problem 4 (10 points):

What is the value of the uncertainty product $(\Delta Y)\cdot(\Delta P_y)$ in the ground state of the 3-dimensional isotropic harmonic oscillator? (3 pts.) Justify your answer. (7 pts.)

Possibly useful numbers and formulae:

$$m(^6\mathrm{Li}) = 6.015\,\mathrm{u}; \qquad 1\,u = 933.5\,\mathrm{MeV}/c^2 = 1.66\times 10^{-27}\,\mathrm{kg}$$

$$\hbar = 6.582\times 10^{-22}\,\mathrm{MeV\,s}, \qquad \hbar c = 197\,\mathrm{MeV\,fm}$$

$$\omega = \sqrt{\frac{k}{m}}$$

$$x_0 = \sqrt{\frac{\hbar}{m\omega}}$$

$$\psi_0(r) = \left(\frac{m\omega}{\pi\hbar}\right)^{3/4} e^{-\frac{m\omega}{2\hbar}r^2} = \frac{1}{(\pi x_0^2)^{3/4}}\, e^{-r^2/(2x_0^2)} \qquad (r^2{=}x^2{+}y^2{+}z^2)$$

$$\int_{-\infty}^{\infty} dx\, e^{-\alpha x^2+\beta x} = e^{\beta^2/(4\alpha)}\sqrt{\frac{\pi}{\alpha}}$$

$$(\Delta\Omega)^2(\Delta\Lambda)^2 \geq \frac{1}{4}\left\{\langle[\hat\Omega-\bar\Omega,\hat\Lambda-\bar\Lambda]_+\rangle^2 + \langle -i[\hat\Omega,\hat\Lambda]_-\rangle^2\right\}$$