Physics 7501 (Quantum Mechanics I):
Homework Set No. 3

Due date: Tuesday, September 15, 2015, 5:00pm
in PRB M2025 (Abhishek Mohapatra's office)

Total point value of set: 100 points

Problem 1 (20 pts.): If $[\hat{A}, \hat{B}] \neq 0$, λ a real or complex scalar, show that

(i) $e^{-\lambda \hat{A}}\, \hat{B}^n\, e^{\lambda \hat{A}} = \left(e^{-\lambda \hat{A}}\, \hat{B}\, e^{\lambda \hat{A}}\right)^n$, (ii) $e^{-\lambda \hat{A}}\, \hat{F}(\hat{B})\, e^{\lambda \hat{A}} = \hat{F}(e^{-\lambda \hat{A}}\, \hat{B}\, e^{\lambda \hat{A}})$,

where n is integer and $\hat{F}(\hat{B})$ is an arbitrary function of $\hat{B}$ that does not depend on any other operators.

Problem 2 (40 pts.): If $[\hat{A}, \hat{B}] = \hat{I}$ ($\hat{I}$ = identity operator), λ a scalar, show

(i) $e^{-\lambda \hat{B}\hat{A}}\, \hat{A}\, e^{\lambda \hat{B}\hat{A}} = e^{\lambda}\, \hat{A}$, (ii) $e^{-\lambda \hat{B}\hat{A}}\, \hat{B}\, e^{\lambda \hat{B}\hat{A}} = e^{-\lambda}\, \hat{B}$,

(iii) $e^{-(\lambda/2)(\hat{B}^2 - \hat{A}^2)}\, \hat{A}\, e^{(\lambda/2)(\hat{B}^2 - \hat{A}^2)} = \hat{A}\, \cosh\lambda + \hat{B}\, \sinh\lambda$,

(iv) $e^{-(\lambda/2)(\hat{B}^2 - \hat{A}^2)}\, \hat{B}\, e^{(\lambda/2)(\hat{B}^2 - \hat{A}^2)} = \hat{B}\, \cosh\lambda + \hat{A}\, \sinh\lambda$.

Problem 3 (30 pts.): Show that

(i) (10 pts.) $\lim_{\epsilon \to 0^+} \delta_\epsilon(x) \equiv \lim_{\epsilon \to 0^+} \left\{ \begin{array}{ll} \dfrac{1}{\epsilon} & \text{for} \quad |x| < \dfrac{\epsilon}{2} \\ 0 & \text{for} \quad |x| > \dfrac{\epsilon}{2} \end{array} \right\} = \delta(x)$,

(ii) (20 pts.) $\lim_{\epsilon \to 0^+} \delta_\epsilon(x) \equiv \lim_{\epsilon \to 0^+} \dfrac{1}{\pi\epsilon} \left(\dfrac{\sin(x/\epsilon)}{x/\epsilon} \right)^2 = \delta(x)$.

In each case you must show that the function in question satisfies, in the limit $\epsilon \to 0$, all the defining properties of the Dirac δ-function (see Shankar, pp. 60-61). You are not allowed to use MATHEMATICA or similar programs for this problem.

Problem 4 (10 pts.): Prove that

(i) $\displaystyle\int_{-\infty}^{\infty} dx\, e^{-\alpha x^2} = \sqrt{\frac{\pi}{\alpha}}$, (ii) $\displaystyle\int_0^{\infty} dx\, x\, e^{-\alpha x^2} = \frac{1}{2\alpha}$,

(iii) $\displaystyle\int_{-\infty}^{\infty} dx\, x^2\, e^{-\alpha x^2} = \frac{1}{2\alpha}\sqrt{\frac{\pi}{\alpha}}$, (iv) $\displaystyle\int_{-\infty}^{\infty} dx\, e^{-\alpha x^2 + \beta x} = e^{\beta^2/(4\alpha)}\sqrt{\frac{\pi}{\alpha}}$.

You cannot use Mathematica for this problem!
Hint for part (i): It may help to first consider a two-dimensional Gaussian integral.