Physics 7501 (Quantum Mechanics I): Homework Set No. 5

Due date: Tuesday, September 29, 2015, 5:00pm in PRB M2025 (Abhishek Mohapatra's office)

Total point value of set: 100 points

Problem 1 (20 pts.): Look at example 4.2.4 in Shankar (p.134–137). Modify it by assuming that the state $|\psi\rangle$ of the system is represented in the basis of position eigenstates by the wave function

$$\langle x|\psi\rangle = \psi(x) = \frac{1}{(\pi\Delta^2)^{1/4}}\, e^{ip_0 x/\hbar}\, e^{-(x-x_0)^2/(2\Delta^2)}.$$

Compute the expectation values and uncertainties of the position and momentum operators, as well as the uncertainty product $\Delta X \cdot \Delta P$, in this state.

Problem 2 (30 pts.):

(i) (10 pts.) A large number of systems are all prepared in the same state $|\psi\rangle$. The observable $\hat{A}$ is measured in all of these systems, with the following result: 50% of the measurements yield the value a_1, 9% of the measurements yield the value a_2, 16% of measurements yield the value a_3, and 25% of the measurements yield the value a_4. Write down the normalized state $|\psi\rangle$.

(ii) (10 pts.) After these measurements, all systems are subjected to a measurement of the observable $\hat{B}$ which is known to be compatible with $\hat{A}$. All measurements yield one of two values for $\hat{B}$, either b_1 or b_2. In the sample of systems in which $\hat{A}$ was measured as a_1, 50% of $\hat{B}$ measurements yield b_1 and 50% yield b_2. In the sample of systems that gave a_2 for the observable $\hat{A}$, all measurements of $\hat{B}$ yield b_1, whereas all systems that gave a_3 for $\hat{A}$ yield the value b_2 when measuring $\hat{B}$. Finally, 1/3 of the systems giving a_4 for $\hat{A}$ yield b_1 for $\hat{B}$, with the rest of the systems yielding b_2 for $\hat{B}$. With this additional knowledge, write down a refined version of the initial state vector $|\psi\rangle$ of all the systems.

(iii) (10 pts.) What are the expectation values of $\hat{A}$ and $\hat{B}$ in the state $|\psi\rangle$? What is the expectation value of $\hat{B}$ in the sample of systems that yielded the value a_4 for $\hat{A}$?

Problem 3 (45 pts.):

A system is prepared in the following superposition of eigenstates of the compatible observables $\hat{\Omega}$ and $\hat{\Lambda}$ with discrete eigenvalues ω_i and λ_j:

$$|\psi\rangle = \frac{1}{2}|\omega_1, \lambda_1\rangle + \frac{1}{2}|\omega_2, \lambda_1\rangle + \frac{1}{\sqrt{2}}|\omega_1, \lambda_2\rangle.$$

(i) (5 pts.) What are the expectation values of $\hat{\Omega}$ and $\hat{\Lambda}$ in this state, respectively?

(ii) (5 pts.) After a measurement of the observable $\hat{\Lambda}$ has yielded the value λ_1 (λ_2), respectively, what is the normalized state of the system in each case?

(iii) (5 pts.) After a measurement of the observable $\hat{\Omega}$ has yielded the value ω_1 (ω_2), respectively, what is the normalized state of the system in each case?

(iv) (5 pts.) Subsequent measurements of $\hat{\Lambda}$ and $\hat{\Omega}$ (in this order) yield the values λ_1 and ω_1. In which normalized state is the system after this pair of measurements? What is the joint probability for this combination of measured values?

(v) (5 pts.) The measurements in part (iv) are made in opposite order, but yield the same measured values for the two observables. Are the joint probabilities and final states the same as in part (iv) or different? Explain your reasoning!

(vi) (10) Repeat the measurements in part (iv), but select those systems whose measurements yield the pair of values (λ_2, ω_1). What is the probability for the outcome λ_2 in the first measurement? Which fraction of all $\hat{\Omega}$ measurements in this measurement protocol yield the value ω_1? How are these probabilities related to the joint probability for the pair of values (λ_2, ω_1) in the combined measurement?

(vii) (10) Repeat the measurements in part (vi) in opposite order (first $\hat{\Omega}$, then $\hat{\Lambda}$) and select again those systems whose measurements yield the pair of values (λ_2, ω_1). What is the probability for the outcome ω_1 in the first measurement? Which fraction of all $\hat{\Lambda}$ measurements in this measurement protocol yield the value λ_2? How are in this situation these two probabilities for the individual measurements related to the joint probability for the pair of values (λ_2, ω_1) in the combined measurement?

Problem 4 (5 pts.):

A system is prepared in the following superposition of eigenstates of an observable $\hat{\Omega}$ which has a continuous spectrum of eigenvalues:

$$|\psi\rangle = \int_{\infty}^{\infty} d\omega \, \psi(\omega) \, |\omega\rangle.$$

Compute the expectation value and uncertainty of the observable $\hat{\Omega}$ in terms of $\psi(\omega)$.