Physics 7501: Homework Set No. 6

Due date: Tuesday, October 20, 2015, 5:00pm in PRB 3018 (Fuyan Lu's office)

Total point value of set: 100 points

The 1-dimensional square well, $V(x) = \begin{cases} -V_0 & |x| \leq a \quad (V_0 > 0) \\ 0 & |x| > a \end{cases}$

(1) $-V_0 \leq E \leq 0$ (bound states) (60 pts.)

(i) Show that the eigenvalue equation for the Hamiltonian $\hat{H} = \frac{\hat{p}^2}{2m} + \hat{V}(\hat{x})$ has alternating even and odd solutions, with energy eigenvalues given by the solutions of

(15)

$$(ka)\tan(ka) = (\kappa a) \qquad \text{(even)}$$
$$(ka)\cot(ka) = -(\kappa a) \qquad \text{(odd)} \qquad (*)$$

where $\hbar\kappa = \sqrt{-2mE}$, $\hbar k = \sqrt{2m(E+V_0)}$. ($\kappa$ = kappa)

(ii) Defining the dimensionless parameter $\beta^2 = \frac{2(ma)(V_0 a)}{\hbar^2}$,

(5) show that $(\kappa a)^2 + (ka)^2 = \beta^2$ and that therefore the solutions for $\chi \equiv ka$ (χ = chi) and for $\left|\frac{E}{V_0}\right|$ depend only on β.

(iii) Solve the eigenvalue equations $(*)$ graphically, by rewriting them as

$$f_{even}(\chi) \equiv \chi^2 \tan^2\chi = \beta^2 - \chi^2 \equiv F(\chi^2),$$
$$f_{odd}(\chi) \equiv \chi^2 \cot^2\chi = F(\chi^2), \qquad (**)$$

(20)

plotting the functions f_{even}, f_{odd}, and F defined in $(**)$ as functions of χ^2, and determining the intersection points between $f_{even/odd}$ and F for $\beta^2 = 30$. From the plot, read off the resulting values for $F \equiv (\kappa a)^2$. For $\beta^2 = 30$, how many bound states do you find?

(iv) Study analytically the limits $V_0 \to \infty$, $a = \text{const.}$ (particle in a box) (5 pts.) and the case of a δ-fct. potential $V(x) = -g\,\delta(x)$,

(20) $g = \lim\limits_{\substack{a \to 0 \\ V_0 \to \infty}} (2aV_0)$. In the latter limit, show that $\beta^2 \to 0$, $k \to \infty$,

$ka \to 0$, and $k^2 a \to mg/\hbar^2$, and that there are no odd solutions and exactly one even solution with energy eigenvalue $E = -\frac{mg^2}{2\hbar^2}$. So the attractive δ-fct. potential supports exactly one bound state.

(2) E > 0 (scattering states) (40 pts.)

(i) Writing $\psi(x) = \begin{cases} A e^{ikx} + B e^{-ikx} & (x < -a) \\ C e^{ik'x} + D e^{-ik'x} & (-a < x < a) \\ F e^{ikx} + G e^{-ikx} & (x > a) \end{cases}$

(20)

show that

$$\begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} \left(\cos(2k'a) - \frac{i\varepsilon'}{2}\sin(2k'a)\right)e^{2ika} & -\frac{i\eta'}{2}\sin(2k'a) \\ \frac{i\eta'}{2}\sin(2k'a) & \left(\cos(2k'a) + \frac{i\varepsilon'}{2}\sin(2k'a)\right)e^{-2ika} \end{pmatrix} \begin{pmatrix} F \\ G \end{pmatrix}$$

where $\varepsilon' = \frac{k'}{k} + \frac{k}{k'}$, $\eta' = \frac{k'}{k} - \frac{k}{k'}$, $\varepsilon'^2 - \eta'^2 = 4$.

(ii) Choosing the boundary condition $G=0$ (no wave incident from the right), compute the transmission coefficient $T = \left|\frac{F}{A}\right|^2$ and
(10) the reflection coefficient $R = \left|\frac{B}{A}\right|^2$ as a function of β.
You should find $T^{-1} = 1 + \frac{1}{4}\eta'^2 \sin^2(2k'a)$.

(iii) Plot T as a function of E/V_0 for a shallow potential with $\beta = 10$ and a deep potential with $\beta = 300$, showing in each case the first five "resonances" where $T = 1$.

(10)

At which values for $(2k'a)$ does T exhibit maxima and minima? Can you interpret these physically? (Hint: think of wavelengths fitting into the potential well.) What are the limiting values of ε', η', and T as $E/V_0 \to \infty$?