Physics 7501 (Quantum Mechanics I): Homework Set No. 9

Due date: Thursday, November 19, 2015, 5:00pm in PRB M2025 (Abhishek Mohapatra's office)

Total point value of set: 100 points

Problem 1 (20 pts.): Exercise 7.4.5 (Shankar, p. 212)

Problem 2 (30 pts.): Exercise 7.5.4 (Shankar, p. 219)

Problem 3 (20 pts.):
(a) (10 pts.) Use the commutation properties of the phonon creation and annihilation operators of the harmonic oscillator, $\hat{a}^\dagger$ and $\hat{a}$, to show that

$$e^{z_1^* \hat{a}}\, e^{z_2 \hat{a}^\dagger} = e^{z_1^* z_2}\, e^{z_2 \hat{a}^\dagger} e^{z_1^* \hat{a}}. \tag{1}$$

Hint: Multiply both sides with $e^{-z_1^* \hat{a}}$ from the right and use the Baker-Campbell-Hausdorff formula. Fill in the missing steps in my lecture notes.
(b) (10 pts.) Using (1), show that the overlap between two coherent states $|z_1\rangle$ and $|z_2\rangle$ satisfies the relation

$$\left|\langle z_1 | z_2 \rangle\right|^2 = e^{-|z_1 - z_2|^2} = e^{-\frac{m\omega}{2\hbar}(x_1 - x_2)^2 - \frac{1}{2m\hbar\omega}(p_1 - p_2)^2} = e^{-[\frac{(p_1-p_2)^2}{2m} + \frac{m\omega^2}{2}(x_1-x_2)^2]/(\hbar\omega)}$$

where (x_1, x_2) are the mean positions and (p_1, p_2) are the mean momenta of the two coherent (aka "classical") states. In which limit do two coherent states become orthogonal?

Problem 4 (30 pts.):

As shown in class (see lecture notes for section 7.6) the propagator for the harmonic oscillator is given in the coherent state representation by the very simple expression

$$U(z_f, t; z_i, 0) = \langle z_f | z_i\, e^{-i\omega t} \rangle = e^{-i\omega t/2} \exp\left[-\frac{|z_i|^2}{2} - \frac{|z_f|^2}{2} + z_f^* z_i e^{-i\omega t} \right]. \tag{2}$$

Use this to compute the propagator $U(x_f, t; x_i, 0)$ in coordinate representation and show that it is given by the expression listed at the bottom of p. 145 of the lecture notes for Lecture 20. [This proves the highly non-trivial identity of the two boxed equations in the bottom half of p. 145 of the lecture notes for Lecture 20.]
Hints: To obtain the correct result it is necessary that you choose the phase of the coherent state wavefunction correctly such that for $t = 0$ it is consistent with Eq. (1), i.e. $\langle z_f | z_i \rangle = \exp\Big[-(|z_i|^2 + |z_f|^2)/2 + z_f^* z_i \Big]$. To do the Gaussian integral you need to "complete the square" in the exponent (i.e. you want to write it in the form $A(x_f, x_i, t) \int_{-\infty}^{\infty} d\eta_1 \ldots d\eta_4 \exp\Big[-\sum_{k,l=1}^4 \eta_k M_{kl} \eta_l \Big]$ where M is a symmetric 4×4 matrix with complex matrix elements, and the η_k are suitably shifted complex integration variables); for a 4-dimensional integral this is tedious, and I recommend to ask Mathematica to do this for you. Once you have completed the square you can use the identity $\int_{-\infty}^{\infty} d\eta_1 \ldots d\eta_4 \exp\Big[-\sum_{k,l=1}^4 \eta_k M_{kl} \eta_l \Big] = \pi^2 / \sqrt{\det(M)}$ to express the result of the Gaussian integral in terms of the determinant of the (symmetric) 4×4 matrix appearing in the exponent of the Gaussian. Again, you can use Mathematica to compute that determinant.