Chapter 11: Symmetries and their consequences

From classical mechanics we know that translational invariance of the Hamilton function $\mathcal{H}$ implies momentum conservation, time-independence of $\mathcal{H}$ implies energy conservation, and rotational invariance of $\mathcal{H}$ implies angular momentum conservation. We now discuss how such symmetries and associated conservation laws manifest themselves in quantum mechanics.

11.1 Translational invariance

Consider single particle in 1 dimension:

Concept	class. mechanics	quantum mechanics
translation	$x \to x+\varepsilon$ $p \to p$	$\langle \hat{X} \rangle \to \langle \hat{X} \rangle + \varepsilon$ $\langle \hat{P} \rangle \to \langle \hat{P} \rangle$
translational invariance	$\mathcal{H} \to \mathcal{H}$	$\langle \hat{H} \rangle \to \langle \hat{H} \rangle$
conservation law	$\frac{dp}{dt} = 0$	$\frac{d}{dt}\langle \hat{P} \rangle = 0$ (to be shown)

Consider an infinitesimal translation ε along the x axis. Under this translation

$\hat{T}$ = translation operator

$$|\psi\rangle \longrightarrow |\psi_\varepsilon\rangle \equiv \hat{T}(\varepsilon)|\psi\rangle$$

What are the properties that $|\psi_\varepsilon\rangle$ must satisfy such that

$$\langle \psi_\varepsilon | \hat{X} | \psi_\varepsilon \rangle = \langle \psi | \hat{X} | \psi \rangle + \varepsilon$$
$$\langle \psi_\varepsilon | \hat{P} | \psi_\varepsilon \rangle = \langle \psi | \hat{P} | \psi \rangle$$

?

Plug in $|\psi_\varepsilon\rangle = T(\varepsilon)|\psi\rangle$:

$$\left.\begin{aligned}\langle\psi|\hat{T}^\dagger(\varepsilon)\hat{X}\hat{T}(\varepsilon)|\psi\rangle &= \langle\psi|\hat{X}|\psi\rangle + \varepsilon \\ \langle\psi|\hat{T}^\dagger(\varepsilon)\hat{P}\hat{T}(\varepsilon)|\psi\rangle &= \langle\psi|\hat{P}|\psi\rangle\end{aligned}\right\}\ \text{for any } |\psi\rangle$$

"active transformation picture"

Since this is supposed to hold for any $|\psi\rangle$, we must have the operator identities

$$\left.\begin{aligned}\hat{X} &\longrightarrow \hat{T}^\dagger(\varepsilon)\hat{X}\hat{T}(\varepsilon) = \hat{X} + \varepsilon\hat{\mathbb{1}} \\ \hat{P} &\longrightarrow \hat{T}^\dagger(\varepsilon)\hat{P}\hat{T}(\varepsilon) = \hat{P}\end{aligned}\right\}$$

"passive transformation picture"

How do we solve these relations? Let us start in the active transformation picture with the action of $\hat{T}(\varepsilon)$ on a position eigenket:

$$\boxed{\hat{T}(\varepsilon)|x\rangle = |x+\varepsilon\rangle}$$

(a particle originally at x must end up at $x+\varepsilon$ after the translation)

($\langle x+\varepsilon|\hat{X}|x+\varepsilon\rangle = (x+\varepsilon)\langle x+\varepsilon|x+\varepsilon\rangle$
$= (x+\varepsilon)\langle x|x\rangle$ (all x-eigenkets have same normalization)
$= \langle x|\hat{X}|x\rangle + \varepsilon$ ✓
We study $\langle x+\varepsilon|\hat{P}|x+\varepsilon\rangle$ below.)

Once we know the action of $\hat{T}(\varepsilon)$ on all basis states, we can compute its action on any state:

$$|\psi_\varepsilon\rangle \equiv \hat{T}(\varepsilon)|\psi\rangle = \hat{T}(\varepsilon)\underbrace{\int_{-\infty}^{\infty} dx\, |x\rangle\langle x|}_{\hat{\mathbb{1}}}\psi\rangle = \int_{-\infty}^{\infty} dx\, \psi(x)|x+\varepsilon\rangle$$

$$\overset{x'=x-\varepsilon}{=} \int_{-\infty}^{\infty} dx'\, \psi(x'-\varepsilon)|x'\rangle$$

$$\Rightarrow \boxed{\psi_\varepsilon(x) = \langle x|\hat{T}(\varepsilon)|\psi\rangle = \psi(x-\varepsilon)}$$

Example: Gaussian wave packet $\psi(x) \sim e^{-x^2/a^2}$ (peaks at $x=0$)

$$\rightarrow \psi(x-\varepsilon) \sim e^{-(x-\varepsilon)^2/a^2} \quad \text{(peaks at } x=a\text{)}$$

What about the invariance of $\langle\hat{P}\rangle$? It is <u>automatic</u>!

$$\langle\psi_\varepsilon|\hat{P}|\psi_\varepsilon\rangle = \int_{-\infty}^{\infty} dx\, \psi_\varepsilon^*(x)\left(-i\hbar\frac{d}{dx}\right)\psi_\varepsilon(x)$$

$$= \int_{-\infty}^{\infty} dx\, \psi^*(x-\varepsilon)\left(-i\hbar\frac{d}{dx}\right)\psi(x-\varepsilon)$$

$$\overset{(x'=x-\varepsilon)}{=} \int_{-\infty}^{\infty} dx'\, \psi^*(x')\left(-i\hbar\frac{d}{dx'}\right)\psi(x') = \langle\psi|\hat{P}|\psi\rangle \checkmark$$

? 6 • How come that in classical physics $p \rightarrow p$ serves as an independent condition, but in quantum mechanics $\langle\hat{P}\rangle \rightarrow \langle\hat{P}\rangle$ appears to be automatic? Well, I cheated!

The condition $\hat{T}(\varepsilon)|x\rangle = |x+\varepsilon\rangle$ is not the most general expression for the action of $\hat{T}(\varepsilon)$ on x-eigenkets. We can more generally demand

$$\hat{T}(\varepsilon)|x\rangle = e^{i\varepsilon g(x)/\hbar}|x+\varepsilon\rangle. \quad (g(x) \text{ arbitrary real function of } x)$$

(Note this is <u>not</u> just a constant phase!)

$$\Rightarrow \langle\psi_\varepsilon|\hat{X}|\psi_\varepsilon\rangle = \langle\psi|\hat{T}^\dagger(\varepsilon)\hat{X}\hat{T}(\varepsilon)|\psi\rangle$$

$$= \int_{-\infty}^{\infty} dx'dx''\, \langle\psi|x'\rangle\langle x'|\hat{T}^\dagger(\varepsilon)\hat{X}\hat{T}(\varepsilon)|x''\rangle\langle x''|\psi\rangle$$

$$= \int_{-\infty}^{\infty} dx'dx''\, \psi^*(x')\underbrace{\langle x'+\varepsilon|e^{-i\varepsilon g(x')/\hbar}\hat{X}e^{i\varepsilon g(x'')/\hbar}|x''+\varepsilon\rangle}_{e^{i\varepsilon(g(x'')-g(x'))/\hbar}(x''+\varepsilon)\underbrace{\langle x'+\varepsilon|x''+\varepsilon\rangle}_{\delta(x'-x'')}}\psi(x'')$$

$$= \int_{-\infty}^{\infty} dx' \, (x'+\varepsilon)\, \psi^*(x')\,\psi(x') \;=\; \langle\psi|\hat{X}|\psi\rangle + \varepsilon \quad \checkmark$$

and

$$\langle\psi_\varepsilon|\hat{P}|\psi_\varepsilon\rangle = \langle\psi|\hat{T}^+(\varepsilon)\,\hat{P}\,\hat{T}(\varepsilon)|\psi\rangle =$$

$$= \int_{-\infty}^{\infty} dx'dx''\, \psi^*(x')\,\langle x'+\varepsilon|\, e^{-i\varepsilon g(x')/\hbar}\,\hat{P}\, e^{i\varepsilon g(x'')}\,|x''+\varepsilon\rangle\,\psi(x'')$$

$$= \int_{-\infty}^{\infty} dx'dx''\, \psi^*(x')\, e^{-i\varepsilon g(x')} \underbrace{\langle x'+\varepsilon|\hat{P}|x''+\varepsilon\rangle}_{\delta(x'-x'')\left(-i\hbar\frac{d}{dx''}\right)} e^{i\varepsilon g(x'')/\hbar}\,\psi(x'')$$

$$= \int_{-\infty}^{\infty} dx'\, \psi^*(x')\left(-i\hbar\frac{d}{dx'}\right)\psi(x') + \int_{-\infty}^{\infty} dx'\, \psi^*(x')\,\psi(x')\, e^{-i\varepsilon g(x')/\hbar}\cdot \underbrace{\left(-i\hbar\frac{d}{dx'}\, e^{i\varepsilon g(x')/\hbar}\right)}_{\varepsilon\,\frac{dg(x')}{dx'}\, e^{i\varepsilon g(x')/\hbar}}$$

$$= \langle\psi|\hat{P}|\psi\rangle + \varepsilon \int_{-\infty}^{\infty} dx'\, \psi^*(x')\,\frac{dg}{dx'}\,\psi(x')$$

$$= \langle\psi|\hat{P}|\psi\rangle + \varepsilon\,\langle\psi|g'(\hat{X})|\psi\rangle \qquad g' = \frac{dg(\hat{X})}{d\hat{X}}$$

Now we need the condition $\langle\hat{P}\rangle \to \langle\hat{P}\rangle$ to eliminate g':

$$\to\; g'(\hat{X}) = 0 \;\Rightarrow\; g(\hat{X}) = \alpha\,\hat{\mathbb{1}} \qquad (\alpha \in \mathbb{R})$$

$$\to\; e^{i\varepsilon g(x)/\hbar}|x+\varepsilon\rangle = e^{i\varepsilon\alpha/\hbar}\,|x+\varepsilon\rangle$$

which is a harmless constant phase factor.

So, while the condition $\langle\hat{X}\rangle \to \langle\hat{X}\rangle + \varepsilon$ allowed for the x-dependent phase $g(x)$, the condition $\langle\hat{P}\rangle \to \langle\hat{P}\rangle$ reduced it to a constant. The same thing will repeat itself when we consider rotations. We will shorten the derivation by anticipating this simplification.

<u>Defn</u>: The Hamiltonian $\hat{H}$ is <u>translationally invariant</u> if $\langle\psi_\varepsilon|\hat{H}|\psi_\varepsilon\rangle = \langle\psi|\hat{H}|\psi\rangle \;\forall |\psi\rangle$.

To work out the implications of translational invariance, we need an explicit expression for $\hat{T}(\varepsilon)$. We proceed as follows: take ε infinitesimal and write

$$\hat{T}(\varepsilon) = \hat{\mathbb{1}} - \frac{i}{\hbar}\varepsilon\hat{G}$$

which is accurate up to linear order in ε.

- $\hat{G}$ = <u>generator</u> of translations
- $\hat{G}^+ = \hat{G}$ (Exercise 11.2.2, homework)

To find $\hat{G}$ plug the ansatz into $\langle x|\hat{T}(\varepsilon)|\psi\rangle = \psi(x-\varepsilon)$:

$$\underbrace{\langle x|\hat{\mathbb{1}}|\psi\rangle}_{\psi(x)} - \frac{i\varepsilon}{\hbar}\langle x|\hat{G}|\psi\rangle + O(\varepsilon^2) = \psi(x) - \varepsilon\frac{d\psi}{dx} + O(\varepsilon^2)$$

$$\Rightarrow \langle x|\hat{G}|\psi\rangle = -i\hbar\frac{d}{dx}\psi(x) = \langle x|\hat{P}|\psi\rangle$$

$$\Rightarrow \boxed{\hat{G} = \hat{P}}$$

$\Rightarrow$ $\boxed{\text{The momentum operator is the generator of translations!}}$

$$\Longleftrightarrow \boxed{\hat{T}(\varepsilon) = \hat{\mathbb{1}} - \varepsilon \frac{i}{\hbar} \hat{P} + \mathcal{O}(\varepsilon^2)}$$

Plugging this into the condition for translation invariance we can derive a conservation law for momentum:

$$\langle \psi | \hat{H} | \psi \rangle = \langle \psi_\varepsilon | \hat{H} | \psi_\varepsilon \rangle = \langle \psi | \hat{T}^+(\varepsilon) \hat{H} \hat{T}(\varepsilon) | \psi \rangle$$

$$= \langle \psi | \left(\hat{\mathbb{1}} + \frac{i\varepsilon}{\hbar} \hat{P}\right) \hat{H} \left(\hat{\mathbb{1}} - \frac{i\varepsilon}{\hbar} \hat{P}\right) | \psi \rangle + \mathcal{O}(\varepsilon^2)$$

$$= \langle \psi | \hat{H} | \psi \rangle + \frac{i\varepsilon}{\hbar} \langle \psi | [\hat{P}, \hat{H}] | \psi \rangle + \mathcal{O}(\varepsilon^2)$$

$$\Rightarrow \quad \langle \psi | [\hat{H}, \hat{P}] | \psi \rangle = 0 \qquad \forall \psi$$

$$\Rightarrow (\text{Ehrenfest}) \quad \boxed{\frac{d}{dt} \langle \hat{P} \rangle = 0} \quad \text{momentum conservation}$$

Rederivation in the passive transformation picture

Before we generalize from infinitesimal to finite translations, let us derive these results in the passive transformation picture:

$$\hat{T}^+(\varepsilon) \hat{X} \hat{T}(\varepsilon) = \hat{X} + \varepsilon \hat{\mathbb{1}}$$

$$\hat{T}^+(\varepsilon) \hat{P} \hat{T}(\varepsilon) = \hat{P}$$

Writing again $\hat{T}(\varepsilon) = \mathbb{1} - \frac{i\varepsilon}{\hbar}\hat{G} + O(\varepsilon^2)$ $(\hat{G}^+ = \hat{G})$

we find

$$(\hat{\mathbb{1}} + \frac{i\varepsilon}{\hbar}\hat{G})\hat{X}(\hat{\mathbb{1}} - \frac{i\varepsilon}{\hbar}\hat{G}) = \hat{X} + \varepsilon\hat{\mathbb{1}} + O(\varepsilon^2)$$

$$\Longleftrightarrow \quad -\frac{i\varepsilon}{\hbar}[\hat{X},\hat{G}] = \varepsilon\hat{\mathbb{1}} \quad \Longleftrightarrow \boxed{[\hat{X},\hat{G}] = i\hbar\hat{\mathbb{1}}}$$

We conclude that $\boxed{\hat{G} = \hat{P} + f(\hat{X})}$

Now we turn to

$$\hat{T}^+(\varepsilon)\hat{P}\hat{T}(\varepsilon) = (\mathbb{1} + \frac{i\varepsilon}{\hbar}\hat{G})\hat{P}(\mathbb{1} - \frac{i\varepsilon}{\hbar}\hat{G}) + O(\varepsilon^2)$$

$$= \hat{P} \quad \Longleftrightarrow -\frac{i\varepsilon}{\hbar}[\hat{G},\hat{P}] = 0 \quad \Rightarrow \boxed{f(\hat{X}) = 0}.$$

Hence $\boxed{\hat{G} = \hat{P}}$

Translational invariance in the passive transformation picture reads $\hat{T}^+(\varepsilon)\hat{H}\hat{T}(\varepsilon) = \hat{H}$

$$\Rightarrow \hat{T}^+(\varepsilon)\hat{H}\hat{T}(\varepsilon) - \hat{H} = 0 \quad \Longleftrightarrow \quad [\hat{H},\hat{T}(\varepsilon)] = O(\varepsilon^2) \xrightarrow[\varepsilon\to 0]{} 0$$

$$\Rightarrow -\frac{i\varepsilon}{\hbar}[\hat{H},\hat{P}] = O(\varepsilon^2) \xrightarrow[\varepsilon\to 0]{} 0$$

$$\Rightarrow \boxed{[\hat{H},\hat{P}] = 0 \overset{\text{Ehrenfest}}{\Longrightarrow} \frac{d}{dt}\langle\hat{P}\rangle = 0}$$

We can also use the general result

$$\hat{U}^\dagger \hat{\Omega}(\hat{X},\hat{P})\hat{U} = \hat{\Omega}(\hat{U}^\dagger\hat{X}\hat{U}, \hat{U}^\dagger\hat{P}\hat{U})$$

(proof: expand Ω into a Taylor series and check term by term)

and the fact that, to linear order in ε, $\hat{T}(\varepsilon)$ is unitary $\left(\hat{T}^\dagger(\varepsilon) = \mathbb{1} + \frac{i\varepsilon}{\hbar}\hat{G} = \hat{T}(-\varepsilon) = (\hat{T}(\varepsilon))^{-1} \right)$

to write the condition for translational invariance as

$$\hat{T}^\dagger(\varepsilon)\hat{H}(\hat{X},\hat{P})\hat{T}(\varepsilon) = \hat{H}\left(\hat{T}^\dagger(\varepsilon)\hat{X}\hat{T}(\varepsilon), \hat{T}^\dagger(\varepsilon)\hat{P}\hat{T}(\varepsilon)\right) + O(\varepsilon^2)$$

$$= \boxed{\hat{H}(\hat{X}+\varepsilon\hat{\mathbb{1}}, \hat{P}) \stackrel{!}{=} \hat{H}(\hat{X},\hat{P})}$$

This looks like the classical condition

$$\mathcal{H}(x,p) = \mathcal{H}(x+\varepsilon, p).$$