Chapter 19: Scattering Theory

One-dimensional scattering: $\lim_{|x|\to\infty} V(x) = 0$.

(1) Consider incoming Gaussian wave packet $\psi(x,t)$ with

$$\lim_{t\to-\infty} \langle \hat{X} \rangle(t) = -\infty \quad \text{and} \quad \langle \hat{P} \rangle = \hbar k_0$$

(2) Expand wavepacket in eigenfunctions of $\hat{H} = \hat{T} + \hat{V}$ for energy $E > 0$

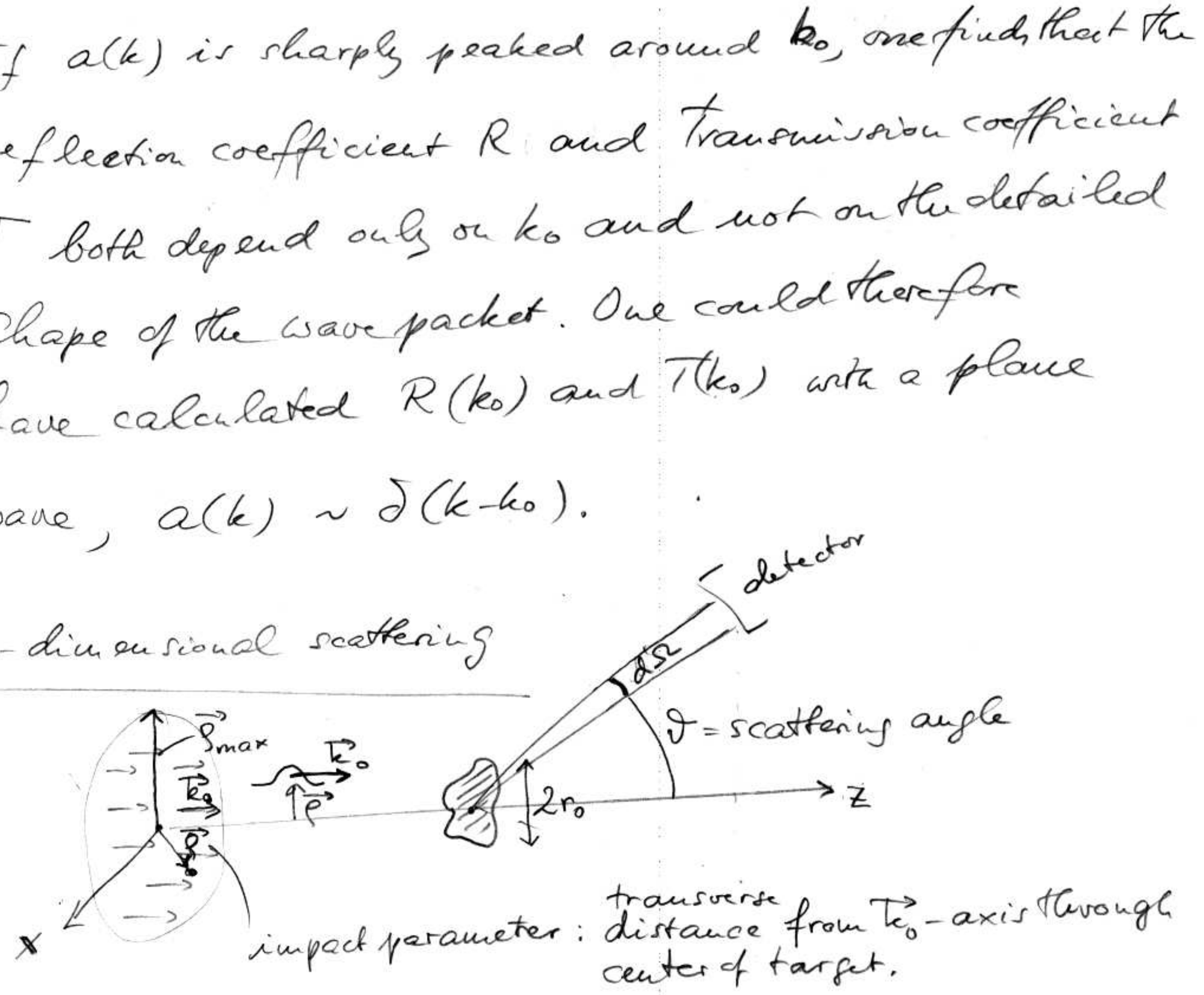
$$\psi(x,t) = \int dk\, a(k)\, e^{-i/\hbar\, E_k t}\, \psi_k(x) \qquad E_k = \frac{\hbar^2 k^2}{2\mu}$$

with

$$\psi_k \begin{cases} \xrightarrow{x\to-\infty} \underset{\text{(incident)}}{A e^{-ikx}} + \underset{\text{(reflected)}}{B e^{ikx}} \\ \xrightarrow{x\to+\infty} \underset{\text{(transmitted)}}{C e^{ikx}}. \end{cases}$$

(3) If $a(k)$ is sharply peaked around k_0, one finds that the reflection coefficient R and transmission coefficient T both depend only on k_0 and not on the detailed shape of the wave packet. One could therefore have calculated $R(k_0)$ and $T(k_0)$ with a plane wave, $a(k) \sim \delta(k-k_0)$.

Three-dimensional scattering



We assume that the distribution of impact parameters of the particles coming out of the accelerator and scattering off the target is uniform over the range r_0 of the interaction potential between projectile and target, $\rho_{max} \gg r_0$. (If $V(r) = e^{-r^2/a^2}$, we say $r_0 = a$ is the "range of the potential".)

The detector sits at an angle ϑ relative to the beam axis $\vec{k}_0$ and at an azimuthal angle ϕ in the $((x,y)=\vec{\rho})$-plane $\perp \vec{k}_0$; it subtends a solid angle $d\Omega = d\varphi \sin\vartheta \, d\vartheta = d\varphi \, d\cos\vartheta$

What we want to compute is the <u>differential cross section</u> $\frac{d\sigma}{d\Omega}$:

$$\frac{d\sigma}{d\Omega}(\vartheta,\varphi)\, d\Omega = \frac{(\#\text{ particles scattered into } d\Omega)/\text{sec}}{(\#\text{ incident particles})/\text{sec}/\text{area in } \vec{\rho}\text{ plane}}$$

We proceed as follows:

(1) Take initial wavepacket with $\langle \hat{\vec{P}} \rangle = \hbar \vec{k}_0$ and mean impact parameter $\langle \vec{\rho} \rangle = \vec{\rho}_0$

(2) Expand it into eigenfunctions of $\hat{H} = \hat{T} + \hat{V}$

$$\psi_{\vec{k}} = \psi_{inc} + \psi_{sc}$$

where $\psi_{inc}(\vec{r}) = e^{i\vec{k}\cdot\vec{r}}$

and $\psi_{sc}(\vec{r})$ is the scattered wave, which is purely outgoing. So $\vec{k}$ labels the incoming wavevector, but only the sum of $\psi_{inc} + \psi_{sc}$ is an eigenfunction of $\hat{H}$,

(In the absence of $\hat{V}$, $\psi_{sc} = 0$.)

(3) Evolve in time:

$$\psi_{\vec{\rho}_0}(\vec{r}, t) = \int d^3k \, a_{\vec{\rho}_0}(\vec{k}) \, e^{-i/\hbar E_{\vec{k}} t} \, \psi_{\vec{k}}(\vec{r})$$

$$E_{\vec{k}} = \frac{\hbar^2 |\vec{k}|^2}{2\mu}$$

time evolution factor

(4) Go to $t \to \infty$ and read off the scattered wave; compute probability current density of ψ_{sc}

Integrate flow of probability into $d\Omega$ at (θ, φ).

One finds that if the momentum space wavefunction $a(\vec{k})$ is sharply peaked around $\vec{k}_0$, this probability depends only on $\vec{k}_0$ and $\langle \vec{\rho} \rangle = \vec{\rho}_0$:

$$P(\vec{\rho}_0; \vec{k}_0 \to d\Omega)$$

(5) If the beam of particles has $\eta(\vec{\rho})$ particles per second per unit area in the $\vec{\rho}$-plane, then the number scattering per second into $d\Omega$ is

$$\eta(d\Omega) = \int \eta(\vec{\rho}) \, P(\vec{\rho}, \vec{k}_0 \to d\Omega) \, d^2\rho$$

In the experiment, we ensure $\eta(\vec{\rho}) = \eta$ independent of $\vec{\rho}$, hence

$$\frac{d\sigma}{d\Omega}(\vartheta,\varphi)\,d\Omega = \frac{\eta(d\Omega)}{\eta} = \int d^2\rho\, P(\vec{\rho}, \vec{k}_0 \to d\Omega)$$

(6) This formula shows that we can get the desired result directly by considering simply a monochromatic wave $\psi_{\vec{k}_0}$ with wave number $\vec{k}_0$, computing in the limit $r \to \infty$ the ratio of probability flow per second into $d\Omega$ associated with the corresponding ψ_{sc} part, to the incident probability current density associated with the $e^{i\vec{k}_0\cdot\vec{r}}$ part.

As we make ψ_{inc} monochromatic with wave # $\vec{k}_0$, it becomes a plane wave which is uniform in $\vec{\rho}$. This makes the integral over $\vec{\rho}$ with $P(\vec{\rho}, \vec{k}_0 \to d\Omega)$ unnecessary. (P is normalized, so as P gets wider in $\vec{\rho}$, its magnitude at any given $\vec{\rho}$ decreases.)

Let's consider step 4) in more detail:

Choose $\vec{e}_z \parallel \vec{k}_0$ and drop subscript 0:

$$\psi_{\vec{k}}(\vec{r}) = e^{ikz} + \psi_{sc}(r,\vartheta,\varphi)$$

(normalization drops out when calculating flux ratio for $\frac{d\sigma}{d\Omega}$).

Far from the scatterer, whose center we place at the origin, the scattered wave solves

$$(\nabla^2 + k^2)\psi_{sc}(r,\vartheta,\varphi) = 0$$ since $rV(r) \xrightarrow[r\to\infty]{} 0$ is assumed.

(We discuss Coulomb scattering later.)

$\Rightarrow$ We can expand

$$\psi_{sc}(r,\vartheta,\varphi) \xrightarrow[r\to\infty]{} \sum_{l,m} \Big(A_l\, j_l(kr) + B_l n_l(kr)\Big) Y_{lm}(\vartheta,\varphi)$$

We want a purely outgoing wave $\sim \frac{e^{ikr}}{kr}$, so we need $A_l / B_l = -i$

$$\Rightarrow \psi_{sc}(r,\vartheta,\varphi) \xrightarrow[r\to\infty]{} \frac{e^{ikr}}{kr} \sum_{lm} (-i)^l (-B_l) Y_{lm}(\vartheta,\varphi)$$
$$\equiv \frac{e^{ikr}}{r} f(\vartheta,\varphi)$$

(using $j_l(kr) \xrightarrow[r\to\infty]{} \frac{1}{kr} \sin\left(kr - \frac{l\pi}{2}\right)$

$n_l(kr) \longrightarrow \frac{-1}{kr} \cos\left(kr - \frac{l\pi}{2}\right)$)

where $\boxed{f(\vartheta,\varphi) = \frac{1}{k} \sum_{lm} (-i)^l (-B_l) Y_{lm}(\vartheta,\varphi)}$

The total wave at $r \to \infty$ looks like

$$\boxed{\psi_{\vec{k}} \xrightarrow[r\to\infty]{} e^{ikz} + f(\vartheta,\varphi)\frac{e^{ikr}}{r}}$$

"scattering amplitude"

Differential cross section:

Need to compute $\vec{j}_{sc}$ and $\vec{j}_{inc}$.

As $r\to\infty$, ψ_{sc} is negligible relative to e^{ikz}, due to its $1/r$ falloff. So we can go to $r\to\infty$ to get the incident current as $\left(\vec{j}=\frac{\hbar}{2\mu i}(\psi^*\vec{\nabla}\psi-\psi\vec{\nabla}\psi^*)\right)$

$$|j_{inc}|=\left|\frac{\hbar}{i}\frac{1}{2\mu}\, e^{-ikz}\vec{\nabla}e^{ikz}-e^{ikz}\vec{\nabla}e^{-ikz}\right|=\frac{\hbar k}{\mu}$$

To get $\vec{j}_{sc}$, we compute

$$\vec{j}_{sc}=\frac{\hbar}{2\mu i}\left(\psi_{sc}^*\vec{\nabla}\psi_{sc}-\psi_{sc}\vec{\nabla}\psi_{sc}^*\right)$$

with $\psi_{sc}=f(\vartheta,\varphi)\frac{e^{ikr}}{r}$. (At any finite angle ϑ, this is the only surviving part of ψ_k if we go to large enough r since any real beam has finite transverse size, so ψ_{inc} is not really infinitely wide in the $\vec{\rho}$-plane)

Use $\vec{\nabla}=\vec{e}_r\frac{\partial}{\partial r}+\vec{e}_\vartheta\frac{1}{r}\frac{\partial}{\partial\vartheta}+\vec{e}_\varphi\frac{1}{r\sin\vartheta}\frac{\partial}{\partial\varphi}$

For $r\to\infty$, $(kr>>>1)$, the two last terms can be ignored relative to the first term:

$$\frac{\partial}{\partial r}\left(f(\vartheta,\varphi)\frac{e^{ikr}}{r}\right)=f(\vartheta,\varphi)\, ik\frac{e^{ikr}}{r}\left(1+\mathcal{O}\left(\frac{1}{kr}\right)\right)$$

$$\Rightarrow \quad \vec{j}_{sc} = \frac{|f(\vartheta,\varphi)|^2}{r^2} \frac{\hbar k}{\mu} \vec{e_r}$$

The probability flow into $d\Omega$ is

$$R(d\Omega) = \vec{j}_{sc} \cdot \vec{e}_r \, r^2 d\Omega \qquad (d\vec{a} = r^2 d\Omega \vec{e}_r)$$

$$= |f|^2 \frac{\hbar k}{\mu} d\Omega$$

The cross section is

$$\frac{d\sigma}{d\Omega} d\Omega = \frac{R(d\Omega)}{j_{inc}} = |f|^2 d\Omega$$

$$\Rightarrow \quad \boxed{\frac{d\sigma}{d\Omega}(\vartheta,\varphi) = |f(\vartheta,\varphi)|^2}$$

Thus, once we know the scattering amplitude $f(\vartheta,\varphi)$, i.e. all the coefficients B_e, we know everything we need.