The Born approximation

(1st order time-dependent perturbation theory)

So the task of scattering theory is essentially to construct, from the knowledge of the time-independent eigenstates of the Hamiltonian including the interaction potential caused by the scatterer, the outgoing spherical wave generated by an incoming plane wave.

In time-dependent perturbation theory, we can formulate this problem in terms of the propagator

$$\hat{S} \equiv \lim_{\substack{t_i \to -\infty \\ t_f \to \infty}} \hat{\mathcal{U}}(t_f, t_i)$$

Its matrix elements between free incoming and outgoing states are called <u>the S-Matrix</u>:

$$P(\vec{p}_i \to d\Omega) = \oint_{\vec{p}_f \, \text{in} \, d\Omega} |\langle \vec{p}_f | \hat{S} | \vec{p}_i \rangle|^2$$

In first-order perturbation theory we can evaluate this using Fermi's golden rule:

$$R^{(1)}_{i \to d\Omega} = \frac{dP(\vec{p}_i \to d\Omega)}{dt} = \frac{2\pi}{\hbar} \int_0^\infty |\langle \vec{p}_f | \hat{V} | \vec{p}_i \rangle|^2 \delta\left(\frac{p_f^2}{2\mu} - \frac{p_i^2}{2\mu}\right) \cdot p_f^2 \, dp_f \, d\Omega$$

it doesn't matter whether we use S or I picture since $|\vec{p}_i\rangle, |\vec{p}_f\rangle$ are $\hat{H}_0$ eigenstates, so $\mathcal{U}_S^0$ generates only a phase

$$= \frac{2\pi}{\hbar} \left| \langle \vec{p}_f | \hat{V} | \vec{p}_i \rangle \right|^2 \mu\, p_i \, d\Omega \qquad (p_f = p_i = \hbar k)$$

This transition rate is the probability flow rate into $d\Omega$

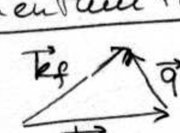
The incoming probability current is

$$\left(|\vec{p}_i\rangle \longleftrightarrow \frac{e^{i \vec{p}_i \cdot \vec{r}/\hbar}}{(2\pi\hbar)^{3/2}} \, , \quad j_{inc} = \rho \cdot v_{inc} \right)$$

$$j_{inc} = \frac{\hbar k}{\mu} \cdot \frac{1}{(2\pi\hbar)^3}$$

$$\Longrightarrow \frac{d\sigma^{(1)}}{d\Omega} = \frac{1}{d\Omega} \frac{R^{(1)}_{i \to d\Omega}}{j_{inc}} = (2\pi)^4 \hbar^2 \mu^2 \left| \langle \vec{p}_f | \hat{V} | \vec{p}_i \rangle \right|^2$$

$$= \left| \frac{\mu}{2\pi\hbar^2} \int d^3 r' \, V(\vec{r}\,') \, e^{-i \vec{q} \cdot \vec{r}\,'} \right|^2 \equiv \left| f^{(1)}(\vartheta, \varphi) \right|^2$$

where $\boxed{\hbar \vec{q} = \vec{p}_f - \vec{p}_i}$ is the momentum transferred to the particle. The direction and magnitude of $\vec{q}$, 

, fix the direction of $\vec{k}_f$ and thus of ϑ, φ:

$$\boxed{\begin{aligned} |\vec{q}|^2 &= |\vec{k}_f - \vec{k}_i|^2 = 2k^2 (1 - \cos\vartheta) = 4k^2 \sin^2(\vartheta/2) \\ \varphi &= \text{azimuthal angle of } \vec{q} \end{aligned}}$$

There is no energy transfer (elastic scattering off an infinitely massive ("nailed down") target).

For a target with finite mass, the result remains the same if we consider $\vec{r}$ the relative coordinate between target and projectile and $E = \frac{\hbar^2 k^2}{2\mu}$ the kinetic energy of the relation motion. In the laboratory frame, the projectile would lose energy due to target recoil, but in the CM frame the relative motion energy is fixed.

We can read off the scattering amplitude:

$$\boxed{f_{Born}(\vartheta,\varphi) \equiv f^{(1)}(\vartheta,\varphi) = -\frac{\mu}{2\pi\hbar^2}\int d^3r'\, V(\vec{r}\,')\, e^{-i\vec{q}\cdot\vec{r}\,'}}$$

The phase factor -1 will only become clear a little later.

So, in this so-called Born approximation, the scattering amplitude $f(\vartheta,\varphi) \equiv f(\vec{q})$ is just the Fourier transform of the scattering potential with respect to the momentum transfer $\vec{q}$.

For central force the scattering potential is spherically symmetric: $V(\vec{r}) = V(r)$. In this case, using a c.o.s. with $\vec{e}_{z'} \parallel \vec{q}$ for the integration,

$$f^{(1)}(\vartheta,\varphi) = -\frac{\mu^2}{2\pi\hbar^2}\int_0^\infty r'^2 dr'\, V(r') \int_0^{2\pi} d\varphi' \int_{-1}^{1} d(\cos\vartheta')\, e^{-iqr'\cos\vartheta'}$$

$$= -\frac{2\mu}{\hbar^2}\int_0^\infty \frac{\sin(qr')}{q} V(r')\, r'dr' \equiv f^{(1)}(\vartheta)$$

the scattering amplitude is azimuthally symmetric and depends only on the (polar) scattering angle ϑ.

Example $\pi - N$ scattering at impact parameter $\rho > r_N \cong 1\,\mathrm{fm}$: For $r > r_N$ the $\pi - N$ potential can be approximated by a Yukawa potential

$$V(r) = \frac{g\, e^{-\mu_\pi r}}{r} \qquad \mu_\pi = \frac{m_\pi c}{\hbar} = \frac{1}{\lambda_\pi}$$

$\uparrow$ Compton wavelength

Using this form for all r, we get

$$f(\vartheta) = -\frac{2\mu g}{q\hbar^2}\int_0^\infty \frac{e^{iqr'} - e^{-iqr'}}{2i}\, e^{-\mu_\pi r'}\, dr'$$

$$= \frac{-2\mu g}{\hbar^2 (q^2 + \mu_\pi^2)}$$

$$\boxed{\frac{d\sigma}{d\Omega} = \frac{4\mu^2 g^2}{\hbar^4 \left[\mu_\pi^2 + 4k^2 \sin^2\frac{\vartheta}{2}\right]^2}}$$

This holds for $\pi - N$ - scattering only for small ϑ. Small-angle scattering is dominated by large impact parameters, whereas scattering at small impact parameters (where our approximation for the potential breaks down) scatters the projectile by large angles.

In the limit $\mu_\pi \to 0$ we get the cross section for

Coulomb scattering: $(g \to Ze^2)$

$$\boxed{\left(\frac{d\sigma}{d\Omega}\right)_{\text{Rutherford}} = \frac{\mu^2 (Ze^2)^2}{4p^4 \sin^4 \vartheta/2} = \frac{(Ze^2)^2}{16 E^2 \sin^4 \vartheta/2}}$$

$E = \frac{p^2}{2\mu} =$ kin. energy of incoming projectile

Rutherford derived this from classical orbits. The quantum mechanical result happens to be the same if all orders of perturbation theory are included, and agrees with the classical result – a double accident! Also, we got the correct result although we used a calculation that assumed $rV(r) \xrightarrow[r\to\infty]{} 0$ which is not true for the Coulomb potential (we approximated the final state Coulomb wave by a plane wave).

Almost too much luck!

The reason why this happened is the following:

Let's take correct Coulomb waves instead of plane waves:

$$\frac{e^{ikr}}{r} \longrightarrow \frac{\widetilde{e^{ikr}}}{r} \equiv \frac{e^{i(kr - \gamma \ln(kr))}}{r} \qquad \left(\gamma = \frac{Ze^2\mu}{\hbar^2 k} = \frac{Ze^2}{\hbar v}\right)$$

"Coulomb parameter"

$$e^{ikz} \longrightarrow \widetilde{e^{ikz}} = e^{ikz + i\gamma \ln(kr - kz)}$$

(Sitenko, p. 63)

One then finds (calculation omitted)

$$\frac{d\sigma}{d\Omega} = |f_c(\vartheta)|^2 \quad \text{with} \quad f_c(\vartheta) = -\frac{\gamma}{2k \sin^2\frac{\theta}{2}} \, e^{-i\left(\gamma \ln(\sin^2\frac{\vartheta}{2}) + \text{const}\right)}$$

const $\in \mathbb{R}$

You see that the Coulomb phase shows up in the scattering amplitude put drops out from the cross section. (In identical-particle scattering, one must (anti-) symmetrize the scattering amplitude, and the Coulomb phase leads to a non-trivial interference term. We will see this later.)

The total Coulomb cross section $\sigma_{\text{Coulomb}} = \int d\Omega \left(\frac{d\sigma}{d\Omega}\right)_{\text{Ruth}}$ diverges, due to an infrared singularity at small scattering angles, resulting from the infinite range of the Coulomb potential which causes particles at any impact parameter (also very large ones) to be deflected. In real life, the Coulomb potential is screened by polarization effects; this cuts off the infrared (small-angle) divergence. You can model the screening by an effective photon mass (Debye-mass) $\mu_D = \frac{m_D c}{\hbar}$, turning the Coulomb into a Yukawa potential. The Yukawa cross section is integrable:

$$\sigma_{\text{Yukawa}} = \int d\Omega \left(\frac{d\sigma}{d\Omega}\right)_{\text{Yukawa}} = \frac{16\pi}{\mu_D^2}\left(\frac{g\,M/\mu_D}{\hbar^2}\right)^2 \frac{1}{1+4k^2/\mu_D^2}$$

$r_D = \frac{1}{\mu_D}$ is the Debye-length or range of the Yukawa potential.

General properties of the scattering amplitude $f^{(1)}(\vartheta)$:

- Let us study the low-energy limit of the scattering amplitude, $k \to 0$:

In this limit, $q = 2k \sin\left(\frac{\vartheta}{2}\right) \to 0$ and

$$f(\vartheta) \longrightarrow -\frac{\mu}{2\pi\hbar^2} \int d^3r' \, V(\vec{r}') \approx -\frac{\mu V_0 r_0^3}{\hbar^2}$$

where r_0 is some effective range of the potential, and V_0 some effective "average" strength of the potential.

- Now the high-energy limit $k \to \infty$:

In this limit $e^{-iqr'\cos\vartheta'}$ oscillates rapidly, i.e. the scattered waves from different points $\vec{r}'$ add with the essentially random phases to zero, except for the small range where the phase is stationary:

$$q \underbrace{r'\cos\vartheta'}_{O(r_0)} \lesssim \pi \;\rightarrow\; 2k \sin\tfrac{\vartheta}{2} \cdot r_0 \lesssim \pi \;\xrightarrow[\substack{\text{since } k \to \infty; \\ \vartheta \to 0}]{}\; k\vartheta r_0 \lesssim \pi$$

$\Rightarrow$ nonzero scattering amplitude only in forward direction $\boxed{\theta \lesssim \frac{1}{kr_0}}$

(This does not work for singular potentials $\sim \frac{1}{r'^3}$, but it works for Coulomb and Yukawa potentials.)

Example Gaussian potential:

$$f(\vartheta) = -\frac{\mu}{2\pi\hbar^2}\int_0^\infty r'^2 dr' \int_0^{2\pi} d\varphi' \int_{-1}^{1} d\cos\vartheta' \, e^{-iqr'\cos\vartheta'} V_0 e^{-r'^2/r_0^2}$$

$$= -\frac{2\mu V_0}{\hbar^2 q}\int_0^\infty r' dr' \sin(qr') e^{-r'^2/r_0^2}$$

$$= -\frac{\mu V_0}{\hbar^2 q}\int_{-\infty}^\infty r' dr' \sin(qr') e^{-r'^2/r_0^2}$$

$$= \frac{-\mu V_0}{2i\hbar^2 q}\int_{-\infty}^\infty e^{-r'^2/r_0^2}\left(e^{iqr'} - e^{-iqr'}\right) r' dr'$$

$$= -\frac{\mu V_0 r_0^2}{2i\hbar^2 q}\int_{-\infty}^\infty \xi \, d\xi \; e^{-\xi^2}\left(e^{iqr_0\xi} - e^{-iqr_0\xi}\right)$$

$$= -\frac{\mu V_0 r_0^2}{2i\hbar^2 q} e^{-\frac{q^2 r_0^2}{4}} \int_{-\infty}^\infty \xi \, d\xi \left(e^{-\left(\xi - \frac{iqr_0}{2}\right)^2} - e^{-\left(\xi + \frac{iqr_0}{2}\right)^2}\right)$$

$$= -\frac{\mu V_0 r_0^2}{2i\hbar^2 q} e^{-\frac{q^2 r_0^2}{4}} \int_{-\infty}^\infty e^{-t^2} dt \left[t + \frac{iqr_0}{2} - \left(t - \frac{iqr_0}{2}\right)\right]$$

$$= -\frac{\mu V_0 r_0^3}{2\hbar^2} e^{-\frac{q^2 r_0^2}{4}} \sqrt{\pi}$$

$$= -\frac{\mu V_0 r_0^3}{2\hbar^2}\sqrt{\pi}\; e^{-k^2 r_0^2 \sin^2 \vartheta/2}$$

This falls off for large angles and becomes negligible for $\sin^2\vartheta/2 \gg \frac{1}{(kr_0)^2}$

For $k \to \infty$, $\sin\vartheta/2 \to \vartheta/2 \to 0$, and $f(\vartheta)$ is non-zero only for $\frac{\vartheta}{2} \lesssim \frac{1}{kr_0}$ ✓