Two-particle scattering

In this section we will discuss the reference frame dependence of the differential cross section for 2-body scattering. Up to now we used (implicitly at least) the CMS as the frame for our discussion. This is not always the best frame to interpret actual experimental data.

If we have only 1 target and it is at rest (i.e. it is not allowed to recoil) then

\[ \# \text{collisions/sec} = \sigma \times \text{incident projectiles/sec/area} \]
\[ = \sigma \cdot p \cdot v \]

where \( v \) is the velocity of the projectile in the beam, and \( p \) is the density of the beam.

Now let's look at beam-beam collisions. Let the other beam have a density \( p_2 \), approaching the projectile at velocity \( v_{\text{rel}} = v_1 + v_2 \) if \( v_1, v_2 \ll c \)

\[ \Rightarrow \# \text{collisions/sec/volume of interaction} \]
\[ = \sigma \cdot p_1 \cdot p_2 \cdot v_{\text{rel}} \]

\( \sigma \) is an area per to the colliding beams, \( \sigma \) is invariant under boosts along the beam axis (i.e. it
is the same as seen from the projectiles', targets', or observer's perspective).

The angular dependence of the differential cross section \( \frac{d\sigma}{d\Omega(\theta, \phi)} \) will, however, depend on the longitudinal reference frame. In the target rest frame, we can define it as

\[
\text{# projectiles scattered into } d(\cos \theta) \, d\phi / \text{sec/ volume} = \frac{d\sigma}{d\Omega_{T}} \, d\Omega_{T} \, s_{1} s_{2} \, v_{\text{rel}}
\]

In this frame, \( v_{\text{rel}} \) is just the projectile velocity, with \( v_{x T} = 0 \), \( \theta_{T}, \phi_{T} \) are the target frame angles.

(We could also define \( \frac{d\sigma}{d\Omega_{T}} \) through the \# targets scattered into \( d\Omega_{T} \), but due to momentum conservation this is uniquely related to what happens to the projectiles, and not an independent measurement.)

In the center-of-mass (CM) frame we define similarly

\[
\text{# projectiles scattered into } d(\cos \theta) \, d\phi / \text{sec/ volume} = \frac{d\sigma}{d\Omega} \, d\Omega \, s_{1} s_{2} \, v_{\text{rel}}
\]

(Variables without subscript denote CM-frame observable.)

In this case \( v_{\text{rel}} = v_{1} + v_{2} \) (nonrelativistically) or \( \frac{v_{1} + v_{2}}{1 + \frac{v_{1} v_{2}}{c^2}} \) (relativistically).
Since $\sigma$ is invariant under longitudinal boosts, we can use the chain rule to relate

$$\frac{d\sigma}{dS^+} = \frac{d\sigma}{dS^2} \frac{dS^2}{dS^+}$$

First, $\frac{d\sigma}{dS^2}$:

Long before the particles begin to interact, we can write their state (in the sense of taking the limit of a very wide wave packet) as

$$\Psi_{\text{inc}}(r_1, r_2) = e^{i \frac{k_1 \cdot r_1}{2}} e^{i \frac{k_2 \cdot r_2}{2}} \quad \text{(in some arbitrary frame)}$$

Choosing $k_1, k_2 \parallel z$, this becomes

$$\Psi_{\text{inc}}(r_1, r_2) = e^{i k_1 \cdot r_1} e^{i k_2 \cdot r_2}$$

where $k_1, k_2$ are the $z$-components of $k_1, k_2$ (i.e., they can be negative) and

$$e^{i (k_1 + k_2) \frac{z_1 + z_2}{2}} e^{i \frac{k_1 - k_2}{2} (z_1 - z_2)} \quad \text{(we assume the particles have equal mass)}$$

The potential depends only on the relative distance, so the CM-motion remains free-particle motion at all times:

$$\Psi(r_1, r_2) = e^{i (k_1 + k_2) \frac{z_1 + z_2}{2}} \Psi_{\text{free}}(r_1, r_2)$$

where $\Psi_{\text{free}}$ is a solution of $\hat{H}_{\text{free}}$. 

(Where $r_1, r_2$ are the relative distances.)
In other words, \( \Psi_{cm} \) has no scattered wave component.

On the other hand, \( \Psi_{me} (\vec{r}_1 - \vec{r}_2) \) will develop a scattered wave:

\[
\Psi_{me} (\vec{r}) = e^{ik\vec{r}} + \Psi_{sc} (\vec{r})
\]

\[
k = \frac{k_1 - k_2}{2}
\]

\[
r \to \infty \quad e^{ik\vec{r}} + f(\theta, \phi) \frac{e^{ikr}}{r}
\]

The time-independent solution for the entire system thus is:

\[
\Psi_{system} (\vec{r}_1, \vec{r}_2) = \Psi_{cm} (\vec{r}_{cm}) \left( e^{ik\vec{r}} + \Psi_{sc} (\vec{r}) \right)
\]

\[
r \to \infty \quad \Psi_{cm} (\vec{r}_{cm}) \left( e^{ik\vec{r}} + f(\theta, \phi) \frac{e^{ikr}}{r} \right)
\]

\[
(r = |\vec{r}_1 - \vec{r}_2|)
\]

In the CM-frame, \( k_1 + k_2 = 0 \), and \( \Psi_{cm} (\vec{r}_{cm}) = 1 \).

So in that frame we can forget about the CM coordinate.

The scattering look in this frame like

The "trajectory" merely serve to illustrate the coordinate, of course, the particle doesn't travel on a classical trajectories.
To find the probability current for the projectile to scatter into \( d\Omega \), we compute the probability current of \( \Psi \) into \( d\Omega \), since \( \vec{p} \) is the same direction as \( \vec{p}_1 \):

\[
\vec{p}_1 \rightarrow d\Omega = \left| f(\theta, \phi) \right|^2 \frac{2\pi k}{\mu} d\Omega |\psi_{cm}|^2 = \text{rate per unit volume of target-beam interactions (see p. 131)}
\]

(The probability density of \( \psi_{cm}(\vec{p}) \) is zero, so \( |\psi_{cm}|^2 \) contributes a factor "1 per unit volume".)

\[
\Rightarrow \frac{d\sigma}{d\Omega} d\Omega = \frac{\vec{p}_1 \rightarrow d\Omega}{\vec{p}_1 \vec{p}_2 \text{v}_{\text{rel}}} = \frac{|f(\theta, \phi)|^2 (\frac{2\pi k}{\mu})}{\vec{p}_1 \vec{p}_2 \text{v}_{\text{rel}}} d\Omega
\]

Now (and we do the following only non-relativistically)

\[
\text{v}_{\text{rel}} = v_1 + v_2 = v_1 \left( 1 + \frac{m_1}{m_2} \right)
\]

\[
= m_1 v_1 \left( \frac{m_1 + m_2}{m_1 m_2} \right)
\]

\[
= \frac{tk}{\mu} = \frac{tk}{\mu} \quad (\text{this is general, but we use it only for } m_1 = m_2)
\]

and \( \vec{p}_1 \vec{p}_2 = |e^{i\mathbf{k} \cdot \mathbf{r}|^2 = |e^{i\mathbf{k} \cdot \mathbf{r}|^2 = 1} \) (per unit volume)

\[
\Rightarrow \frac{d\sigma}{d\Omega} = \left| f(\theta, \phi) \right|^2
\]

So what we have been calculating all along can be re-interpreted as the differential cross section in the CM frame!

What about the target rest frame?
For this we need to compute \( d\sigma \frac{d\Omega}{d\Omega_{\text{rest}}} \).
\[ \tan \theta = \frac{\sqrt{P_x^2 + P_y^2}}{P_z} - \frac{P_x}{P_z} \]

\[ \tan \phi = \frac{P_y}{P_x} \]

\[ \tan \theta_T = \frac{P_x}{P_z + \sqrt{P_x^2 + P_y^2}} \quad \text{\( (P_x, P_y \text{ invariant under boosts}) \)} \]

We see that \( \phi_T = \phi \) and \( \tan \theta_T = \frac{P_x}{P_z + \sqrt{P_x^2 + P_y^2}} = \frac{\sin \theta}{\cos \theta + 1} = \tan \theta_T \).

\[ \Rightarrow \theta_T = \frac{\theta}{2} \quad \text{\( \text{for equal mass particles} \)} \]

\[ \Rightarrow \theta_T \leq \frac{\pi}{2} \text{ always!} \]

\[ \frac{d\sigma}{d\Omega_T} (\theta) = \frac{d\sigma}{d\Omega} (\theta = 2\theta_T) 4\cos \theta_T \quad \text{(homework)} \]
Scattering of identical particles

Consider scattering of two identical spin-0 bosons in their CM frame. These must be described by a symmetrized wave function \( \Psi_{\text{ch}} = \frac{1}{2}(\Psi_1 + \Psi_2) \) is invariant under particle exchange while \( \Psi = \Psi_1 - \Psi_2 \) does not hold. So, while \( \Psi_{\text{ch}}^* (\vec{r}_1^0) \) is automatically symmetric, we must symmetrize \( \Psi_{\text{el}} (\vec{r}) \) by hand:

\[
\Psi_{\text{el}}^\text{sym} (\vec{r}) \rightarrow \frac{e^{i k r}}{r} \left( e^{i k r} + e^{-i k r} \right) + \left[ f(\theta, \phi) + f(\pi - \theta, \phi + \pi) \right] \frac{e^{i k r}}{r}.
\]

(The normalization drops out in the cross section when we divide by the incident flux.)

\[
\Rightarrow f_{\text{sym}} (\theta, \phi) = f(\theta, \phi) + f(\pi - \theta, \phi + \pi)
\]

(We cannot say which of the two identical particles scatters into \( (\theta, \phi) \) and which into \( (\pi - \theta, \phi + \pi) \):

\[
\begin{array}{c}
\longrightarrow \quad \text{vs.} \quad \longrightarrow \\
\end{array}
\]

\[
\Rightarrow \frac{d\sigma}{dS^2} = \left| f(\theta, \phi) + f(\pi - \theta, \phi + \pi) \right|^2 = \left| f(\theta, \phi) \right|^2 + \left| f(\pi - \theta, \phi + \pi) \right|^2 + 2\Re[f(\theta, \phi)f^*(\pi - \theta, \phi + \pi)]
\]

First 2 terms: Prob. for scattering 2 distinguishable particles, with one or the other ending up inside.

Note: (1) To find \( \sigma \), we must integrate only \( \theta \) over \( 2\pi \) radians (not \( 4\pi \)), to avoid double counting
(2) The accident that you get the correct Rutherford cross section by treating the incoming and outgoing Coulomb waves as plane waves does not repeat itself now: the phase of \( f(\Omega, \mathbf{p}) \) matters in the interference term. The correct cross section for Coulomb scattering of identical \( \pi^+ \) mesons (spin zero, charge 1) is

\[
\frac{d\sigma}{d\Omega} = \left( \frac{e^2}{4E} \right)^2 \left[ \frac{1}{\sin^4 \left( \frac{\Omega}{2} \right)} + \frac{1}{\cos^4 \left( \frac{\Omega}{2} \right)} + \frac{2 \cos \left( \gamma \right) \ln \left( \tan^2 \left( \frac{\Omega}{2} \right) \right)}{\sin^2 \left( \frac{\Omega}{2} \right) \cos^2 \left( \frac{\Omega}{2} \right)} \right]
\]

\( (\gamma = \frac{Ze^2}{\hbar \nu}) \)

Whereas the trick of taking the Yukawa cross section for \( \mu \rightarrow 0 \) (which assures plane wave initial and final states) would give a different interference term, and Rutherford’s classical approach would give no interference term at all.

For identical spin-\( \frac{1}{2} \) fermions, let us study scattering by a spin-independent interaction. The 2 fermions can be in a spin-singlet state, which has a symmetric spatial wave function, and a spin-triplet state with an antisymmetric spatial wave function. If the electron beam is unpolarized, \( S_2 \) is randomly distributed, a triplet state, are 3 times as likely as singlet states:

\[
\frac{d\sigma}{d\Omega} = \frac{3}{4} \left| f(\Omega, \mathbf{p}) - f(\pi - \delta, \mathbf{p} + \mathbf{x}) \right|^2 + \frac{1}{4} \left| f(\Omega, \mathbf{p}) + f(\pi - \delta, \mathbf{p} + \mathbf{x}) \right|^2
\]

\[= \left( \frac{e^2}{4E} \right)^2 \left[ \frac{1}{\sin^4 \left( \frac{\Omega}{2} \right)} + \frac{1}{\cos^4 \left( \frac{\Omega}{2} \right)} + \frac{2 \cos \left( \gamma \right) \ln \left( \tan^2 \left( \frac{\Omega}{2} \right) \right)}{\sin^2 \left( \frac{\Omega}{2} \right) \cos^2 \left( \frac{\Omega}{2} \right)} \right]\]