Two-particle scattering

In this section we will discuss the reference frame dependence of the differential cross section for 2-body scattering. Up to now we used (implicitly at least) the CMS as the frame for our discussion. This is not always the best frame to interpret actual experimental data.

If we have only 1 target and it is at rest (i.e. it is not allowed to recoil) then

$$\# \text{collisions/sec} = \sigma \times \text{incident projectiles/sec/area}$$
$$= \sigma \cdot \rho_1 v_1$$

where v_1 is the velocity of the projectiles in the beam, and ρ_1 is the density of the beam.

Now let's look at beam-beam collisions. Let the other beam have a density ρ_2, approaching the projectiles at velocity $v_{rel}\left(= v_1 + v_2 \text{ if } v_1, v_2 \ll c\right)$

$\Rightarrow$ # collisions/sec/volume of interaction

$$= \sigma \rho_1 \rho_2 v_{rel}$$

σ is an area $\perp$ to the colliding beams, so it is invariant under boosts along the beam axis (i.e. it is the same as seen from the projectiles', targets', or CM observer's perspective).

The angular dependence of the differential cross section $\frac{d\sigma}{d\Omega}(\vartheta,\varphi)$ will, however, depend on the longitudinal reference frame. In the target rest frame, we can define it as

$$\#\text{projectiles scattered into } d(\cos\vartheta_T)\,d\phi_T/\text{sec}/\text{volume}$$

$$= \frac{d\sigma}{d\Omega_T}\, d\Omega_T\, \rho_1 \rho_2 v_{rel}$$

In this frame, v_{rel} is just the projectile velocity, with $v_{2T}=0$. ϑ_T, ϕ_T are the target frame angles.

(We could also define $\frac{d\sigma}{d\Omega_T}$ through the # targets scattered into $d\Omega_T$, but due to momentum conservation this is uniquely related to what happens to the projectiles, and not an independent measurement.)

In the center-of-mass (CM) frame we define similarly

$$\#\text{projectiles scattered into } d(\cos\vartheta)\,d\phi/\text{sec}/\text{volume} =$$

$$= \frac{d\sigma}{d\Omega}\, d\Omega\, \rho_1 \rho_2 v_{rel}$$

(variables without subscript denote CM-frame observables).

In this case $v_{rel} = v_1 + v_2$ (nonrelativistically) or $\frac{v_1+v_2}{1+\frac{v_1 v_2}{c^2}}$ (relativistically)

Since σ is invariant under longitudinal boosts,

we can use the chain rule to relate

$$\frac{d\sigma}{d\Omega_T} = \frac{d\sigma}{d\Omega}\frac{d\Omega}{d\Omega_T}$$

First, $\frac{d\sigma}{d\Omega}$:

Long before the two particles begin to interact, we can write their state (in the sense of taking the limit of a very wide wavepacket) as

$$\psi_{inc}(\vec{r}_1, \vec{r}_2) = e^{i\vec{k}_1\cdot\vec{r}_1} e^{i\vec{k}_2\cdot\vec{r}_2} \qquad \text{(in some arbitrary frame)}$$

Choosing $\vec{k}_1, \vec{k}_2 \parallel z$, this becomes

$$\psi_{inc}(\vec{r}_1, \vec{r}_2) = e^{ik_1 z_1} e^{ik_2 z_2}$$

where k_1, k_2 are the z-components of $\vec{k}_1, \vec{k}_2$ (i.e. they can be negative)

$$= e^{i(k_1+k_2)\frac{z_1+z_2}{2}} e^{i\frac{k_1-k_2}{2}(z_1-z_2)}$$

(we assume the particles have equal mass)

$$= \psi^{CM}_{inc}(z_{CM})\, \psi^{rel}_{inc}(z) \qquad z_{CM} = \tfrac{1}{2}(z_1+z_2),\ z = z_1 - z_2$$

The potential depends only on the relative distance, so the CM-motion remains free-particle motion at all times and everywhere:

$$\psi(\vec{r}_1, \vec{r}_2) = e^{i(k_1+k_2)z_{CM}}\, \psi_{rel}(\vec{r}_1 - \vec{r}_2)$$

where ψ_{rel} is a solution of $\hat{H}_{rel}$.

In other words, ψ_{CM} has no scattered wave component.

On the other hand, $\psi_{rel}(\vec{r} = \vec{r}_1 - \vec{r}_2)$ will develop a scattered wave:

$$\psi_{rel}(\vec{r}) = e^{ikz} + \psi_{sc}(\vec{r}) \qquad k = \frac{k_1 - k_2}{2}$$

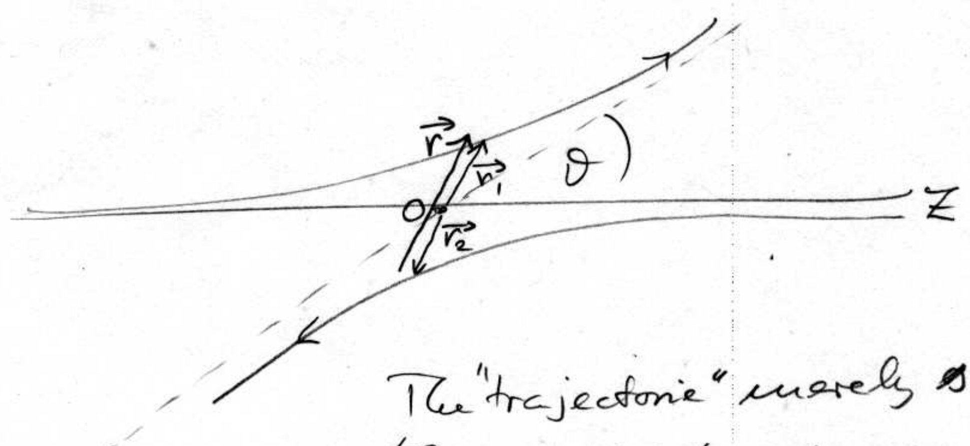
$$\xrightarrow[r\to\infty]{} e^{ikz} + f(\vartheta,\varphi)\frac{e^{ikr}}{r}.$$

The time-independent solution for the entire system thus is

$$\psi_{system}(\vec{r}_1,\vec{r}_2) = \psi^{CM}(z_{CM})\left(e^{ikz} + \psi_{sc}(\vec{r})\right)$$

$$\xrightarrow[r\to\infty]{} \psi^{CM}(z_{CM})\left(e^{ikz} + f(\vartheta,\varphi)\frac{e^{ikr}}{r}\right) \qquad \left(r = |\vec{r}_1 - \vec{r}_2|\right)$$

In the CM frame, $k_1 + k_2 = 0$, and $\psi^{CM}(z_{CM}) = 1$.
So in that frame we can forget about the CM coordinate.
The scattering look in this frame like



The "trajectorie" merely serve to illustrate the coordinates; of course, the particle doesn't travel on a classical trajectory.

To find the probability current for the projectile to scatter into $d\Omega$, we compute the probability current of $\psi_{sc}(\vec{r})$ into $d\Omega$, since $\vec{r}$ is the same direction as $\vec{r_1}$:

$$R_{i\to d\Omega} = |f(\vartheta,\varphi)|^2 \frac{\hbar k}{\mu} d\Omega |\psi^{CM}|^2 = \text{rate per unit volume of target-beam interactions}$$

$$= \frac{d\sigma}{d\Omega} d\Omega \rho_1 \rho_2 v_{rel} \quad (\text{see p } 131)$$

(The probability density of $\psi^{CM}(z)$ is $\underline{\text{unity in the CM frame}}$ so $|\psi^{CM}|^2$ contributes a factor "1 per unit volume".)

$$\Rightarrow \frac{d\sigma}{d\Omega} d\Omega = \frac{R_{i\to d\Omega}}{\rho_1 \rho_2 v_{rel}} = \frac{|f(\vartheta,\varphi)|^2 (\hbar k/\mu)}{\rho_1 \rho_2 v_{rel}} d\Omega$$

Now (and we do the following only non-relativistically)

$$v_{rel} = v_1 + v_2 = v_1\left(1 + \frac{m_1}{m_2}\right) \qquad (m_1 v_1 = m_2 v_2)$$

$$= m_1 v_1 \left(\frac{m_1 + m_2}{m_1 m_2}\right)$$

(this is general, but we use it only for $m_1 = m_2$.)

$$= \hbar k \frac{1}{\mu} = \frac{\hbar k}{\mu}$$

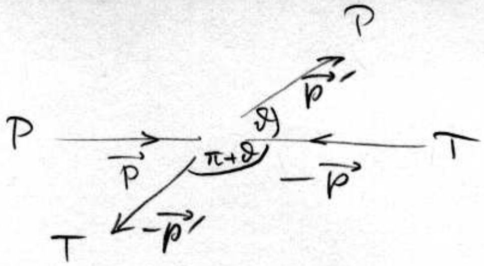
and $\rho_1 = \rho_2 = |e^{ik_1 z}|^2 = |e^{ik_2 z}|^2 = 1$ (per unit volume)

$$\Rightarrow \boxed{\frac{d\sigma}{d\Omega} = |f(\vartheta,\varphi)|^2}$$

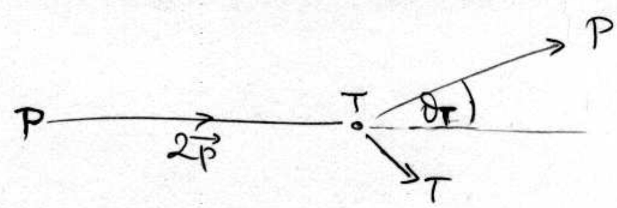
So what we have been calculating all along can be re-interpreted as the differential cross section in the CM frame!

What about the target rest frame?

For this we need to compute $\frac{d\Omega}{d\Omega_T}$.



CM frame



target-rest frame

<u>CM frame</u>

$$\tan\vartheta = \frac{\sqrt{p_x'^2 + p_y'^2}}{p_z'} = \frac{p_\perp'}{p_z'} \qquad (x, y \perp \text{beam})$$

$$\tan\varphi = \frac{p_y'}{p_x'}$$

<u>target frame</u>

$$\tan\vartheta_T = \frac{p_\perp'}{\underbrace{p_z' + p}_{p_{zT}'}} \qquad \left(\begin{array}{l} p_\perp' \text{ is invariant under boosts,} \\ p_{zT}' \text{ differs by } p \text{ from } p_z' \end{array}\right)$$

$$\tan\varphi_T = \frac{p_y'}{p_x'} \qquad (p_x', p_y' \text{ invariant under boosts})$$

We see that $\varphi = \varphi_T$ and

$$\tan\vartheta_T = \frac{p_\perp'/p}{p_z'/p + 1} = \frac{p_\perp'/p'}{p_z'/p' + 1} = \frac{\sin\vartheta}{\cos\vartheta + 1} = \tan(\vartheta/2)$$

$$\Rightarrow \quad \vartheta_T = \frac{\vartheta}{2} \qquad (\text{for equal mass particles})$$

$$\Rightarrow \begin{cases} \vartheta_T \le \frac{\pi}{2} \quad \text{always!} \\ \frac{d\sigma}{d\Omega_T}(\vartheta_T) = \frac{d\sigma}{d\Omega}(\vartheta = 2\vartheta_T)\, 4\cos\vartheta_T \quad (\text{homework}) \end{cases}$$

Scattering of identical particles

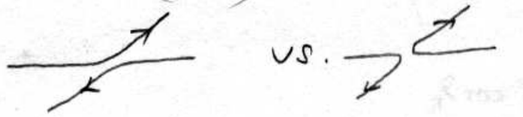
Consider scattering of two identical spin-0 bosons in their CM frame. These must be described by a symmetrized wave function. $\vec{r}_{CM} = \frac{1}{2}(\vec{r}_1 + \vec{r}_2)$ is invariant under particle exchange while $\vec{r} = \vec{r}_1 - \vec{r}_2$ changes sign. So, while $\psi^{CM}(\vec{r}_{CM})$ is automatically symmetric, we must symmetrize $\psi_{rel}(\vec{r})$ by hand:

$$\psi_{rel}^{sym}(\vec{r}) \xrightarrow[r\to\infty]{} \left(e^{ikz} + e^{-ikz}\right) + \left[f(\vartheta,\varphi) + f(\pi-\vartheta, \phi+\pi)\right]\frac{e^{ikr}}{r}$$

(The normalization drops out in the cross section when we divide by the incident flux.)

$$\Rightarrow f_{sym}(\vartheta,\varphi) = f(\vartheta,\varphi) + f(\pi-\vartheta, \phi+\pi)$$

(We cannot say which of the two identical particles scatters in (ϑ,ϕ) and which into $(\pi-\vartheta, \phi+\pi)$:



)

$$\Rightarrow \frac{d\sigma}{d\Omega} = \left|f(\vartheta,\varphi) + f(\pi-\vartheta, \phi+\pi)\right|^2 = \left|f(\vartheta,\varphi)\right|^2 + \left|f(\pi-\vartheta,\phi+\pi)\right|^2 + 2\,\mathrm{Re}\left(f(\vartheta,\varphi)f^*(\pi-\vartheta, \phi+\pi)\right)$$

First 2 terms: prob. for scattering 2 distinguishable particles, with one <u>or</u> the other ending up in $d\Omega$.

<u>Note</u>: (1) To find σ, we must integrate only over 2π radians (not 4π), to avoid double counting

(2) The accident that you get the correct Rutherford cross section by treating the incoming and outgoing Coulomb waves as plane waves does not repeat itself now: the phase of $f(\vartheta,\varphi)$ matters in the interference term. The correct cross section for Coulomb scattering of identical π^+ mesons (spin zero, charge e) is

$$\frac{d\sigma}{d\Omega} = \left(\frac{e^2}{4E}\right)^2 \left[\frac{1}{\sin^4(\frac{\vartheta}{2})} + \frac{1}{\cos^4(\vartheta/2)} + \frac{2\cos(\gamma \ln(\tan^2\frac{\vartheta}{2}))}{\sin^2\frac{\vartheta}{2}\cos^2\vartheta/2}\right]$$

$$\left(\gamma = \frac{Ze^2}{\hbar v}\right)$$

whereas the trick of taking the Yukawa cross section for $\mu_0 \to 0$ (which assumes plane wave initial and final states) would give a different interference term, and Rutherford's classical approach would give no interference term at all.

For identical spin-½ fermions (say electrons), let us study scattering by a spin-independent interaction. The 2 fermions can be in a spin-singlet state, which has a symmetric spatial wavefunction, and a spin-triplet state with antisymmetric spatial wavefunction. If the electron beam is unpolarized, S_z is randomly distributed, a triplet states are 3 times as likely as singlet states:

$$\frac{d\sigma}{d\Omega} = \frac{3}{4}\left|f(\vartheta,\varphi) - f(\pi-\vartheta,\varphi+\pi)\right|^2 + \frac{1}{4}\left|f(\vartheta,\varphi) + f(\pi-\vartheta,\varphi+\pi)\right|^2$$

$$= \left(\frac{e^2}{4E}\right)^2\left[\frac{1}{\sin^4(\vartheta/2)} + \frac{1}{\cos^4(\vartheta/2)} - \frac{\cos(\gamma \ln(\tan^2\frac{\vartheta}{2}))}{\sin^2(\vartheta/2)\cos^2(\vartheta/2)}\right]$$