Chapter 12: Rotational Invariance and Angular Momentum

Nontrivial rotations exist only in more than 1 dimensions.

To warm up we will first handle the case of 2 dimensions.

This is enough to discuss rotational invariance for problems with cylindrical symmetry. Only afterwards we will discuss rotational invariance in 3d (spherical symmetry).

12.1 Translations in higher dimensional spaces

As a background we quickly review translations in more than 1d (the 2- and 3-d transls. are then special cases).

By a straightforward extension of the methods in chapter 11 we find the generator for a translation in direction $\hat{\vec{n}}$:

$$\hat{G}_{\hat{n}} = \hat{P}_{\hat{n}} = \hat{\vec{n}} \cdot \hat{\vec{P}} \qquad \left(\text{e.g. translation in } z\text{-direction: } \hat{n} = (0,0,1),\ \hat{G}_z = \hat{P}_z \xrightarrow[x\text{-basis}]{} -i\hbar \frac{\partial}{\partial z}\right)$$

So infinitesimal translations by a vector $\vec{\varepsilon} = (\varepsilon_x, \varepsilon_y, \varepsilon_z)$ (in $\mathbb{R}^3$) or $\vec{\varepsilon} = (\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n)$ (in $\mathbb{R}^n$) ($|\vec{\varepsilon}| \ll 1$) are described by the translation operator

$$\boxed{\hat{T}(\vec{\varepsilon}) = \mathbb{1} - \frac{i}{\hbar} \vec{\varepsilon} \cdot \hat{\vec{P}} + O(|\vec{\varepsilon}|^2) = \hat{\mathbb{1}} - \frac{i}{\hbar}\left(\varepsilon_x \hat{P}_x + \varepsilon_y \hat{P}_y + \varepsilon_z \hat{P}_z\right) + O(|\vec{\varepsilon}|^2)}$$

and finite translations by a vector $\vec{a}$ by

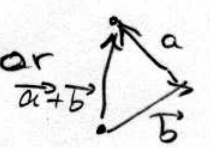
$$\boxed{\hat{T}(\vec{a}) = e^{-\frac{i}{\hbar} \vec{a} \cdot \hat{\vec{P}}} = e^{-\frac{i}{\hbar}(a_x \hat{P}_x + a_y \hat{P}_y + a_z \hat{P}_z + \dots)} = e^{-\frac{i}{\hbar} a \hat{P}_{\hat{a}}}}$$

where $a \equiv |\vec{a}|$, and $\hat{P}_{\hat{a}} = \hat{a} \cdot \hat{\vec{P}}$ is in the x-basis represented by $-i\hbar$ times the derivative in the direction of $\vec{a}$.

Since the different components of $\hat{\vec{P}}$ commute and $\vec{a}$ is a constant vector, the order of terms in the exponent doesn't matter. Also, this implies that

$$\boxed{\hat{T}(\vec{a})\hat{T}(\vec{b}) = e^{-\frac{i}{\hbar}\vec{a}\cdot\hat{\vec{P}}} e^{-i/\hbar\, \vec{b}\cdot\hat{\vec{P}}} = e^{-\frac{i}{\hbar}(\vec{a}+\vec{b})\cdot\hat{\vec{P}}} = \hat{T}(\vec{a}+\vec{b})}$$

i.e. that a translation by $\vec{b}$ ↗ followed by a translation by $\vec{a}$ ↖ corresponds to a translation by the sum

vector 

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12.2. Rotations in two dimensions

Consider a rotation $\hat{R}(\varphi_0 \vec{e}_z)$ (i.e. a rotation by angle φ_0 around the z-axis, for positive φ_0 counterclockwise in the x-y plane):

$$\begin{pmatrix} x \\ y \end{pmatrix} \longrightarrow \hat{R}\begin{pmatrix} x \\ y \end{pmatrix} \equiv \begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix} = \begin{pmatrix} \cos\varphi_0 & -\sin\varphi_0 \\ \sin\varphi_0 & \cos\varphi_0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \qquad (*)$$

$$\begin{pmatrix} p_x \\ p_y \end{pmatrix} \longrightarrow \hat{R}\begin{pmatrix} p_x \\ p_y \end{pmatrix} \equiv \begin{pmatrix} \bar{p}_x \\ \bar{p}_y \end{pmatrix} = \begin{pmatrix} \cos\varphi_0 & -\sin\varphi_0 \\ \sin\varphi_0 & \cos\varphi_0 \end{pmatrix}\begin{pmatrix} p_x \\ p_y \end{pmatrix}$$

(the same transf. applies to <u>all</u> vectors in the x-y plane).

For any vector in the x-y plane, $\hat{R}(\varphi_0 \vec{e}_z)$ is represented by the matrix $R(\varphi_0 \vec{e}_z) = \begin{pmatrix} \cos\varphi_0 & -\sin\varphi_0 \\ \sin\varphi_0 & \cos\varphi_0 \end{pmatrix}$

Now consider the effect of such a rotation on a quantum mechanical state $|\psi\rangle$. We denote it as

$$|\psi\rangle \xrightarrow[\hat{R}(\varphi_0 \vec{e}_z)]{} |\psi_R\rangle = \hat{\mathcal{U}}[R]\,|\psi\rangle$$

where $\hat{\mathcal{U}}$ is some unitary operator (such that the norm of $|\psi\rangle$ is preserved, $\langle\psi_R|\psi_R\rangle = \langle\psi|\,\hat{\mathcal{U}}^{+}(R)\,\hat{\mathcal{U}}(R)|\psi\rangle \overset{!}{=} \langle\psi|\psi\rangle = 1.$)

$\hat{\mathcal{U}}$ has to be structured such that equations $(*)$ carry over to the expectation values of $\hat{\vec{X}}$ and $\hat{\vec{P}}$ in an arbitrary state $|\psi\rangle$:

$$\left.\begin{aligned}
&(\underline{\underline{1}}) \quad \langle \hat{X} \rangle_R = \langle \hat{X} \rangle \cos\varphi_0 - \langle \hat{Y} \rangle \sin\varphi_0 \\
&(\underline{\underline{2}}) \quad \langle Y \rangle_R = \langle \hat{X} \rangle \sin\varphi_0 + \langle \hat{Y} \rangle \sin\varphi_0 \\
&(\underline{\underline{3}}) \quad \langle \hat{P}_x \rangle_R = \langle \hat{P}_x \rangle \cos\varphi_0 - \langle \hat{P}_y \rangle \sin\varphi_0 \\
&(\underline{\underline{4}}) \quad \langle P_y \rangle_R = \langle \hat{P}_x \rangle \sin\varphi_0 + \langle \hat{P}_y \rangle \cos\varphi_0
\end{aligned}\right\}$$

where

$\langle \hat{X} \rangle_R = \langle \psi_R | \hat{X} | \psi_R \rangle$ etc.

$\langle \hat{X} \rangle = \langle \psi | \hat{X} | \psi \rangle$ etc.

To figure out $\hat{\mathcal{U}}[R]$ acts on a generic $|\psi\rangle$, let's first study its action on position eigenstates:

$$\hat{\mathcal{U}}[R]\,|x,y\rangle = |x\cos\varphi_0 - y\sin\varphi_0,\ x\sin\varphi_0 + y\cos\varphi_0\rangle \qquad (✡)$$

(As for translations, we should allow for an (x,y)-dependent phase factor $g(x,y)$ which we then show to collapse to a constant phase by virtue of the constraints $(\underline{\underline{3}})$ and $(\underline{\underline{4}})$. I will omit that step.)

To construct $\hat{\mathcal{U}}[R]$, let's first consider infinitesimal rotations $\varepsilon_z \vec{e}_z$ $(\varphi_0 \to \varepsilon_z \ll 1)$ and write

$$\hat{\mathcal{U}}[\hat{R}(\varepsilon_z \vec{e}_z)] = \hat{\mathbb{1}} - \frac{i}{\hbar}\varepsilon_z \hat{L}_z + O(\varepsilon_z^2)$$

where $\hat{L}_z = \hat{L}_z^+$ is the <u>generator of infinitesimal rotations</u> around the z-axis. What is $\hat{L}_z$? Plug this ansatz into (✡) and expand the r.h.s. in a Taylor series in ε_z up to linear order:

$$\left(\hat{\mathbb{1}} - \frac{i}{\hbar}\varepsilon_z \hat{L}_z + O(\varepsilon_z^2)\right)|x,y\rangle = |x - y\varepsilon_z + O(\varepsilon_z^2), x\varepsilon_z + y + O(\varepsilon_z^2)\rangle$$

Use this in the general expansion

$$|\psi\rangle = \int_{-\infty}^{\infty} dx\,dy\; C(x,y)\,|x,y\rangle$$

and project $|\psi_R\rangle = \hat{\mathcal{U}}[R]|\psi\rangle$ onto $\langle x,y|$:

$$\psi_R(x,y) = \langle x,y|\hat{\mathbb{1}} - \frac{i}{\hbar}\varepsilon_z \hat{L}_z|\psi\rangle + O(\varepsilon_z^2) = \psi(x + \varepsilon_z y, y - \varepsilon_z x) + O(\varepsilon_z^2)$$

$$= \psi(x,y) + y\varepsilon_z \frac{\partial\psi}{\partial x}(x,y) - \varepsilon_z x \frac{\partial\psi}{\partial y}(x,y) + O(\varepsilon_z^2)$$

The terms of $O(\varepsilon_z^0)$ cancel, and the linear terms give

$$\langle x,y|\hat{L}_z|\psi\rangle = \left[x\left(-i\hbar\frac{\partial}{\partial y}\right) - y\left(-i\hbar\frac{\partial}{\partial x}\right)\right]\psi(x,y)$$

$$\equiv \left[x(-i\hbar\partial_y) - y(-i\hbar\partial_x)\right]\psi(x,y) \quad \text{(shorthand notn.)}$$

So we see that in the coordinate basis $\hat{L}_z$ is represented by

$$\hat{L}_z \underset{x,y\text{-basis}}{\longleftrightarrow} x(-i\hbar\partial_y) - y(-i\hbar\partial_x)$$

from which we conclude for the operator itself that

$$\boxed{\hat{L}_z = \hat{X}\hat{P}_y - \hat{Y}\hat{P}_x}$$

In the momentum basis

$$\hat{L}_z \xleftrightarrow[p_x,p_y\text{-basis}]{} (i\hbar\partial_{p_x})p_y - (i\hbar\partial_{p_y})p_x$$

and

$$-\frac{i}{\hbar}\varepsilon_z \langle p_x,p_y|\hat{L}_z|\psi\rangle = (p_y\varepsilon_z)\frac{\partial\psi(p_x,p_y)}{\partial p_x} - (p_x\varepsilon_z)\frac{\partial\psi(p_x,p_y)}{\partial p_y}$$

Such that

$$\langle p_x,p_y|\psi_R\rangle = \psi_R(p_x,p_y) = \psi(p_x+\varepsilon_z p_y, p_y-\varepsilon_z p_x)$$

From $\psi_R(x,y) = \psi(x+\varepsilon_z y, y-\varepsilon_z x)$

$\psi_R(p_x,p_y) = \psi(p_x+\varepsilon_z p_y, p_y-\varepsilon_z p_x)$

a straightforward calculation gives Eqs. (1) - (4).

In the passive transformation picture one has instead the constraints

(1′) $\hat{U}^+[R]\hat{X}\hat{U}[R] = \hat{X} - \varepsilon_z\hat{Y}$

(2′) $\hat{U}^+[R]\hat{Y}\hat{U}[R] = \hat{Y} + \varepsilon_z\hat{X}$

(3′) $\hat{U}^+[R]\hat{P}_x\hat{U}[R] = \hat{P}_x - \varepsilon_z\hat{P}_y$

(4′) $\hat{U}^+[R]\hat{P}_y\hat{U}[R] = \hat{P}_y + \varepsilon_z\hat{P}_x$

Plugging in the ansatz $\hat{U} = \hat{\mathbb{1}} - \frac{i}{\hbar}\varepsilon_z\hat{L}_z$ gives

$[\hat{X},\hat{L}_z] = -i\hbar\hat{Y}$ $\qquad$ $[\hat{P}_x,\hat{L}_z] = -i\hbar\hat{P}_y$

$[\hat{Y},\hat{L}_z] = i\hbar\hat{X}$ $\qquad$ $[\hat{P}_y,\hat{L}_z] = i\hbar\hat{P}_x$