These commutation relations are sufficient to show that $\hat{L}_z = \hat{X}\hat{P}_y - \hat{Y}\hat{P}_x$.

Finite rotations

$$\boxed{\hat{U}[\hat{R}(\varphi_0\vec{e}_z)] = \lim_{N\to\infty}\left(\mathbb{1} - \frac{i}{\hbar}\overset{\overset{\varepsilon_z}{\downarrow}}{\frac{\varphi_0}{N}}\hat{L}_z\right)^N = e^{-i/\hbar\,\varphi_0\hat{L}_z}}$$

Representation in polar coordinates:

$$x = \rho\cos\varphi \quad , \; y = \rho\sin\varphi$$

We want that under rotation by φ_0 around z-axis,

$\psi(\rho,\varphi) \longrightarrow \psi(\rho,\varphi-\varphi_0)$, or (infinitesimally)

$\psi(\rho,\varphi) \longrightarrow \psi(\rho,\varphi-\varepsilon_z)$

$$\Rightarrow \quad \hat{L}_z \leftrightarrow -i\hbar\frac{\partial}{\partial\varphi} \quad \text{such that} \quad -\frac{i}{\hbar}\varepsilon_z\hat{L}_z \leftrightarrow -\varepsilon_z\frac{\partial}{\partial\varphi}$$

This is consistent with what we derived since

$$\frac{\partial}{\partial\varphi}\psi(x,y) = \frac{\partial}{\partial\varphi}\psi(\rho\cos\varphi,\rho\sin\varphi) = \frac{\partial\psi}{\partial x}\frac{\partial(\rho\cos\varphi)}{\partial\varphi} + \frac{\partial\psi}{\partial y}\frac{\partial(\rho\sin\varphi)}{\partial\varphi}$$

$$= -\rho\sin\varphi\frac{\partial\psi}{\partial x} + (\rho\cos\varphi)\frac{\partial\psi}{\partial y}$$

$$= (-y\partial_x + x\partial_y)\psi$$

$$\Rightarrow \; -i\hbar\partial_\varphi\psi(x,y) = \Big(x(-i\hbar\partial_y) - y(-i\hbar\partial_x)\Big)\psi \; \checkmark$$

Finite rotations are then given by

$$\hat{U}[\vec{R}(\varphi_0\vec{e}_z)] = e^{-i/\hbar\,\varphi_0\hat{L}_z} \longleftrightarrow e^{-\frac{i}{\hbar}\varphi_0\left(-i\hbar\frac{\partial}{\partial\varphi}\right)} = e^{-\varphi_0\frac{\partial}{\partial\varphi}}$$

which represents nothing but the Taylor expansion in φ:

$$\psi(\rho, \varphi - \varphi_0) = e^{-\varphi_0 \frac{\partial}{\partial\varphi}} \psi(\rho,\varphi) = \left(1 - \varphi_0 \frac{\partial}{\partial\varphi} + \frac{\varphi_0^2}{2!}\frac{\partial^2}{\partial\varphi^2} - \frac{\varphi_0^3}{3!}\frac{\partial^3}{\partial\varphi^3} \dots\right)\psi(\rho,\varphi)$$

Obviously

$$\hat{\mathcal{U}}[\hat{R}(\varphi_0 \vec{e}_z)]\, \hat{\mathcal{U}}[\hat{R}(\varphi_1 \vec{e}_z)] = e^{-i/\hbar\, \varphi_0 \hat{L}_z}\, e^{-i/\hbar\, \varphi_1 \hat{L}_z}$$

$$= e^{-i/\hbar (\varphi_0 + \varphi_1) \hat{L}_z} = \hat{\mathcal{U}}[\hat{R}((\varphi_0+\varphi_1)\vec{e}_z)] \checkmark$$

How should we think about $\hat{L}_z$ physically?

Well, $\hat{L}_z = \hat{X}\hat{P}_y - \hat{Y}\hat{P}_x = (\hat{\vec{R}} \times \hat{\vec{P}})_z$

(no problem with non-commutativity since in cross product there are no terms $\sim \hat{X}_i \hat{P}_i$, $(i=1,2,3)$, only terms $\hat{X}_i \hat{P}_j$, $i \neq j$, which commute.)

So $\hat{L}_z$ is the operator describing the z-component of orbital angular momentum $\vec{L} = \vec{r} \times \vec{p}$.

Angular momentum conservation:

A problem has rotational symmetry around the z-axis (is "rotational invariant") if

$$\langle \psi_R | \hat{H} | \psi_R \rangle = \langle \psi | \hat{H} | \psi \rangle \text{ or}$$

$$\hat{\mathcal{U}}^+[R]\, \hat{H}(\hat{X}, \hat{P}_x; \hat{Y}, \hat{P}_y)\, \hat{\mathcal{U}}[R] = \hat{H}[\hat{X}, \hat{P}_x; \hat{Y}, \hat{P}_y]$$

infinitesimal rotn. $\Rightarrow$ $$[\hat{H}, \hat{L}_z] = 0$$

Ehrenfest $\Rightarrow$ $$\boxed{\frac{d}{dt}\langle L_z \rangle = 0}$$ conservation of z-comp. of angular momentum.

For rotationally invariant problems, $\hat{H}$ and $\hat{L}_z$ have a common eigenbasis.

If we write $\hat{H}$ as a function of $\hat{R} = \sqrt{\hat{X}^2 + \hat{Y}^2}$, and $\hat{\Phi} = \tan^{-1}\left(\frac{\hat{Y}}{\hat{X}}\right)$, then rotational invariance implies

$$\hat{U}^+ \hat{H}(\hat{R}, \hat{\Phi}) \hat{U} = \hat{H}(\hat{U}^+\hat{R}\hat{U}, \hat{U}^+\hat{\Phi}\hat{U})$$
$$= \hat{H}(\hat{R}, \hat{\Phi} - \varphi_0 \hat{\mathbb{1}}) \stackrel{!}{=} \hat{H}(\hat{R}, \hat{\Phi})$$

i.e. $\hat{\Phi}$-independence of $\hat{H}$.

Combining translations and rotations

Since the generators for translations and rotations don't commute,

$$[\hat{P}_x, \hat{L}_z] = -i\hbar \hat{P}_y \quad , [\hat{P}_y, \hat{L}_z] = i\hbar \hat{P}_x$$

the net effect an infinitesimal translation, say, in x, followed by an infinitesimal rotation, followed by the reverse translation in x, followed by the reverse rotation, is a net infinitesimal translation in y, etc.

See example on p. 311 in the text. Try with sheet on paper for finite rots. + translations.

12.3 The eigenvalue problem of $\hat{L}_z$

For rotationally invariant problems, a common set of eigenfunctions of $\hat{H}$ and $\hat{L}_z$ can be found. To construct it, we first have to solve for the eigenstates of $\hat{L}_z$:

$$\hat{L}_z |l_z\rangle = l_z |l_z\rangle$$

In coordinate basis

$$-i\hbar \frac{\partial \psi_{l_z}(\rho,\varphi)}{\partial \varphi} = l_z \psi_{l_z}(\rho,\varphi)$$

Solution:
$$\psi_{l_z}(\rho,\varphi) = R(\rho)\, e^{i l_z \varphi/\hbar}$$

where is $R(\rho)$ is arbitrary as long as it is normalizable:

$$\int dx\,dy\, |R(\rho)|^2 = \int_0^{2\pi} d\varphi \int_0^{\infty} \rho\, d\rho\, R^2(\rho) = 1 .$$

Since $\hat{L}_z = \hat{X}\hat{P}_y - \hat{Y}\hat{P}_x$ is Hermitean, l_z must be real.

Since $0 \le \varphi \le 2\pi$ and the wave function should be unique,

we need $\psi(\rho, 2\pi) = \psi(\rho, 0)$

$$\Rightarrow e^{i l_z 2\pi/\hbar} = 1 \quad \Rightarrow \quad \boxed{l_z = m\hbar, \quad m = 0, \pm 1, \pm 2, \ldots}$$

"magnetic quantum number"

(Other values of l_z lead to wavefunctions outside the Hilbert space on which $\hat{L}_z$ is Hermitean.)

What about the function $R(\rho)$? Here we have an infinite freedom: for each m, the Hilbert space of $\mathbb{V}_m$ of normalizable states with $l_z = m\hbar$ is infinite dimensional. To further classify these eigenstates, we can choose another Hermitean operator that commutes with $\hat{L}_z$ and whose simultaneous eigenfunctions with $\hat{L}_z$ pick out a basis in $\mathbb{V}_m$. For a rotationally invariant problem, $\hat{H}$ is such an operator. We will see that its eigenstates pick out a unique basis in $\mathbb{V}_m$.

$\Rightarrow$ l_z not enough to completely classify the eigenstates, but (l_z, E) are sufficient for a rotationally invariant problem in 2d.

Introduce

$$\langle \varphi | m \rangle \equiv \Phi_m(\varphi) = \frac{e^{im\varphi}}{\sqrt{2\pi}}$$

As far as the φ-dependence goes, these are non-degenerate eigenfunctions of $\hat{L}_z$. They are <u>orthonormal</u>:

$$\int_0^{2\pi} \Phi_m^*(\varphi)\, \Phi_{m'}(\varphi) = \langle m | \underbrace{\left(\int_0^{2\pi} \frac{d\varphi}{2\pi} |\varphi\rangle\langle\varphi| \right)}_{\hat{\mathbb{1}}} | m' \rangle = \delta_{mm'}$$

Solutions to rotationally invariant problems in 2d:

Consider a problem where $V(\rho,\varphi) = V(\rho)$ is φ-independent.

In the coordinate-representation the eigenvalue eqn. of the Hamiltonian reads

$$\Big[\underbrace{-\frac{\hbar^2}{2\mu}\Big(\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial\varphi^2}\Big)}_{\Leftrightarrow \frac{\hat{p}^2}{2\mu}} + V(\rho)\Big]\psi_E(\rho,\varphi) = E\,\psi_E(\rho,\varphi) \quad (*)$$

use μ for mass, to avoid confusion with $l_z = m\hbar$

where we rewrote $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial\varphi^2}$ in cylindrical coordinates. We seek solutions that are simultaneously eigenfunctions of $\hat{L}_z$, with, say, eigenvalue $m\hbar$:

$$\psi_{Em}(\rho,\varphi) = R_{Em}(\rho)\,\frac{e^{im\varphi}}{\sqrt{2\pi}} = R_{Em}(\rho)\,\Phi_m(\varphi)$$

Plugging this ansatz into $(*)$ gives

$$\Big[-\frac{\hbar^2}{2\mu}\Big(\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial}{\partial\rho} - \frac{m^2}{\rho^2}\Big) + V(\rho)\Big] R_{Em}(\rho) = E\,R_{Em}(\rho)$$

where we deleted a factor $\Phi_m(\varphi)$ from both sides.

The solutions R_{Em} and the eigenvalues E depend on $V(\rho)$; the eigenfunctions $\Phi_m(\varphi)$ of $\hat{L}_z$, however, remain the same for all $V(\rho)$.

$\rightarrow$ $\Phi_m(\varphi) = \langle\varphi|m\rangle$ provide angular part of the wavefunction in the eigenvalue problem of <u>all</u> rotationally invariant problems.

<u>Example</u> The isotropic 2-dimensional oscillator (Exercise 12.3.7, p. 316)

$$\hat{H} = \frac{\hat{P}_x^2 + \hat{P}_y^2}{2\mu} + \frac{\mu}{2}\omega^2(\hat{X}^2 + \hat{Y}^2)$$

- First, check rotational invariance around z-axis:

$$[\hat{H}, \hat{L}_z] = [\hat{H}, \hat{Y}\hat{P}_x - \hat{X}\hat{P}_y] = \frac{1}{2\mu}\Big([\hat{P}_x^2, -\hat{X}]\hat{P}_y + [\hat{P}_y^2, \hat{Y}]\hat{P}_x\Big)$$
$$+ \frac{\mu\omega^2}{2}\Big(\hat{Y}[\hat{X}^2, \hat{P}_x] - \hat{X}[\hat{Y}^2, \hat{P}_y]\Big)$$

$$= \frac{1}{2\mu}\left(\hat{P}_x(2i\hbar)\hat{P}_y - 2i\hbar\hat{P}_y\hat{P}_x\right) + \frac{\mu\omega^2}{2}\left(\hat{Y}(2i\hbar)\hat{X} - \hat{X}(2i\hbar)\hat{Y}\right)$$

$$= 0 \checkmark$$

Hence $\hat{H}$ and $\hat{L}_z$ share a common set of eigenstates.

- We know the eigenstates $\hat{L}_z$: $\hat{L}_z|\Phi_m\rangle = m\hbar|\Phi_m\rangle$
  with coord. repr. $\Phi_m(\varphi) = \frac{e^{im\varphi}}{\sqrt{2\pi}}$

$$\Rightarrow \hat{L}_z(\hat{H}|\Phi_m\rangle) = \hat{H}\hat{L}_z|\Phi_m\rangle = m\hbar\,\hat{H}|\Phi_m\rangle \Rightarrow \hat{H}|\Phi_m\rangle \sim |\Phi_m\rangle$$

We can expand a general state $|\psi\rangle$ as

$$|\psi\rangle = \sum_E \sum_{m=-\infty}^{\infty} C_{Em}|Em\rangle$$

where $|Em\rangle$ are common eigenstates of $\hat{H}$ and $\hat{L}_z$:

$$\hat{L}_z|Em\rangle = m\hbar|Em\rangle$$
$$\hat{H}|Em\rangle = E|Em\rangle$$

The states $|Em\rangle$ can be represented in the position eigenbasis $|\rho\varphi\rangle = |\rho\rangle \otimes |\varphi\rangle$ as follows:

$$|Em\rangle \xrightarrow{\rho\varphi\text{ basis}} \psi_{Em}(\rho, \varphi) = \langle\rho\varphi|Em\rangle$$

The eigenvalue equation for $\hat{H}$ in this basis reads

$$\left[-\frac{\hbar^2}{2\mu}\underbrace{\left(\underbrace{\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial}{\partial\rho}}_{\frac{\partial^2}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial}{\partial\rho}} + \frac{1}{\rho^2}\frac{\partial^2}{\partial\varphi^2}\right)}_{\Delta \text{ in 2d}} + \frac{\mu\omega^2}{2}\rho^2\right]\psi_{Em}(\rho,\varphi) = E\psi_{Em}(\rho,\varphi)$$

Making a separation of variables ansatz, $\psi_{Em}(\rho,\varphi) = R_{Em}(\rho)\Phi_m(\varphi)$, an noting that we already know the solution for Φ_m, this reduces to

$$\frac{1}{\rho}\frac{d}{d\rho}\rho\frac{d}{d\rho}R_{Em}(\rho) + \left[\frac{2\mu}{\hbar^2}\left(E - \frac{\mu\omega^2}{2}\rho^2\right) - \frac{m^2}{\rho^2}\right]R_{Em}(\rho) = 0$$

or

$$\boxed{\frac{d^2R}{d\rho^2} + \frac{1}{\rho}\frac{dR}{d\rho} + \left(\frac{2\mu E}{\hbar^2} - \frac{\mu^2\omega^2}{\hbar^2}\rho^2 - \frac{m^2}{\rho^2}\right)R = 0}$$

reduced radial problem

- Next let's find the natural length and energy scales for this problem: Write $\xi = \rho/\rho_0$:

$$\frac{d^2R}{d\xi^2} + \frac{1}{\xi}\frac{dR}{d\xi} + \left(\frac{2\mu E\rho_0^2}{\hbar^2} - \frac{\mu^2\omega^2\rho_0^4}{\hbar^2}\xi^2 - \frac{m^2}{\xi^2}\right)R = 0$$

Choose $\frac{\mu^2\omega^2\rho_0^4}{\hbar^2} = 1 \Rightarrow \rho_0^4 = \frac{\hbar^2}{\mu^2\omega^2} \Rightarrow \boxed{\rho_0 = \sqrt{\frac{\hbar}{\mu\omega}}}$

natural length scale, similar to 1-d oscillator

Next $2\varepsilon \equiv \frac{2\mu E\rho_0^2}{\hbar^2} = \frac{2\mu E}{\hbar^2}\frac{\hbar}{\mu\omega} = 2\frac{E}{\hbar\omega} \Rightarrow \hbar\omega$ = natural energy scale = phonon energy

$$\Rightarrow \boxed{\frac{d^2R}{d\xi^2} + \frac{1}{\xi}\frac{dR}{d\xi} + \left(2\varepsilon - \xi^2 - \frac{m^2}{\xi^2}\right)R = 0}$$

- Now study asymptotics for $\xi \to \infty$ and $\xi \to 0$:

- For $\xi \to 0$: $\quad \frac{d^2R}{d\xi^2} + \frac{1}{\xi}\frac{dR}{d\xi} - \frac{m^2}{\xi^2}R = 0$

Ansatz: $\quad R(\xi) = \xi^{\alpha}\sum_{k=0}^{\infty} c_k \xi^k = \sum_{k=0}^{\infty} c_k \xi^{k+\alpha}$

Plug in: $\quad \sum_{k=0}^{\infty}\Big[(k+\alpha)(k+\alpha-1)\,c_k\,\xi^{k+\alpha-2} + (k+\alpha)c_k\,\xi^{k+\alpha-2} - m^2 c_k\,\xi^{k+\alpha-2}\Big] = 0 \quad \forall \xi$

For $k=0$ this gives $\quad \alpha(\alpha-1) + \alpha - m^2 = 0$

$$\Rightarrow \boxed{\alpha = \pm|m|}$$

Now, we want

$$\langle Em|Em\rangle = 1 = \int_0^{\infty} \rho\, d\rho\; R^2_{Em}(\rho) = 1$$

$\Rightarrow$ $R(\rho \to 0)$ can not diverge faster than $\frac{1}{\rho^{1-\varepsilon}}$, with $\varepsilon > 0$

$$\Rightarrow \boxed{\alpha = |m|\,, \quad m = 0, \pm1, \pm2, \dots}$$

$$\Rightarrow \boxed{R(\rho) \xrightarrow[\rho\to 0]{} \rho^{|m|}\,, \quad m = 0, \pm1, \pm2, \dots}$$

- For $\xi \to \infty$: $\quad \frac{d^2R}{d\xi^2} - \xi^2 R \approx 0$

$$\Rightarrow \quad R(\xi) \xrightarrow[\xi\to\infty]{} e^{-\xi^2/2} \quad \text{up to powers of } \xi$$

$$\Rightarrow \boxed{R(\rho) \xrightarrow[\rho\to\infty]{} e^{-\frac{\mu\omega}{2\hbar}\rho^2}} \quad \text{up to powers of } \rho$$

These two asymptotic limits lead us to make the ansatz

$$R_{Em}(\xi) = \xi^{|m|} e^{-\xi^2/2} \mathcal{U}_{Em}(\xi)$$

- Plug this into the differential equation for R to obtain a simpler equation for $\mathcal{U}$:

Following the hint, we first write $R = \xi^{|m|} f$, then plug in $f = e^{-\xi^2/2}\mathcal{U}$ in a second step.

$$\frac{dR}{d\xi} = |m|\,\xi^{|m|-1} f + \xi^{|m|} f'$$

$$\frac{d^2R}{d\xi^2} = |m|(|m|-1)\,\xi^{|m|-2} f + 2|m|\,\xi^{|m|-1} f' + \xi^{|m|} f''$$

$$\frac{d^2R}{d\xi^2} + \frac{1}{\xi}\frac{dR}{d\xi} + \left(2\varepsilon - \xi^2 - \frac{m^2}{\xi^2}\right)R =$$

$$= \left[\frac{m^2}{\xi^2} + \left(2\varepsilon - \xi^2 - \frac{m^2}{\xi^2}\right)\right]\xi^{|m|} f + \frac{2|m|+1}{\xi}\xi^{|m|} f' + \xi^{|m|} f'' = 0$$

$$\Rightarrow \boxed{f'' + \frac{2|m|+1}{\xi} f' + (2\varepsilon - \xi^2) f = 0}$$ intermediate result

$$f' = -\xi e^{-\xi^2/2}\mathcal{U} + e^{-\xi^2/2}\mathcal{U}'$$

$$f'' = -e^{-\xi^2/2}\mathcal{U} + \xi^2 e^{-\xi^2/2}\mathcal{U} - 2\xi e^{-\xi^2/2}\mathcal{U}' + e^{-\xi^2/2}\mathcal{U}''$$

Cancel factor $e^{-\xi^2/2}$:

$$(\xi^2-1)\mathcal{U} - 2\xi\mathcal{U}' + \mathcal{U}'' - (2|m|+1)\mathcal{U} + \frac{2|m|+1}{\xi}\mathcal{U}'$$

$$+ (2\varepsilon - \xi^2)\mathcal{U} = 0$$

$$\boxed{\mathcal{U}'' + \left(\frac{2|m|+1}{\xi} - 2\xi\right)\mathcal{U}' + 2(\varepsilon - |m| - 1)\mathcal{U} = 0}$$

- Now, since we have already extracted the leading factors at large and small ξ, $\mathcal{U}(\xi)$ should be expandable in a power series:

$$\mathcal{U}(\xi) = \sum_{r=0}^{\infty} C_r \xi^r \qquad \text{with } C_0 \neq 0 \quad (\text{otherwise } R \text{ does not approach } \xi^{|m|} \text{ for } \xi \to 0)$$

Inserting this into the differential equation for $\mathcal{U}$ gives

$$\sum_{r=0}^{\infty} \Big[\underbrace{C_r\, r(r-1)\,\xi^{r-2} + C_r\, r\,(2|m|+1)\,\xi^{r-2}}_{C_r\,(r^2+2|m|r)\,\xi^{r-2}} - 2\,C_r\, r\,\xi^r + 2(\varepsilon-|m|-1)\,C_r\xi^r \Big] = 0$$

Relabel the summation index for the first term:

$$\sum_{r=0}^{\infty} \Big[C_{r+2}\big((r+2)^2 + 2|m|(r+2)\big) - 2(r+|m|+1-\varepsilon)\,C_r \Big]\,\xi^r = 0 \quad \forall \xi$$

$$\Rightarrow \boxed{C_{r+2} = 2\,\frac{r+|m|+1-\varepsilon}{(r+2)(r+2|m|)}\,C_r} \quad \text{recursion relation}$$

For $r \to \infty$ this recursion relation gives

$$\frac{C_{r+2}}{C_r} \xrightarrow[r\to\infty]{} \frac{2}{r} \quad \Rightarrow \quad \mathcal{U}(\xi) \xrightarrow[\xi\to\infty]{} e^{\xi^2} = \sum \frac{\xi^{2n}}{n!}$$

$$\left(r = 2n \quad \frac{C_{r+2}}{C_r} = \frac{C_{n+1}}{C_n} = \frac{1}{n} = \frac{2}{r}\right)$$

But this would make $R = \xi^{|m|} e^{-\xi^2/2}\,\mathcal{U} \sim \xi^{|m|} e^{+\xi^2/2}$ non-renormalizable !

$\Rightarrow$ The recursion relation must break off!

This happens if

$$\boxed{\varepsilon = r + |m| + 1 \quad \text{for some } r}$$

Since $C_0 \neq 0$, the truncation of the even series requires r to be even. But for even r, the odd series never truncates unless $C_1 = 0$.

So we have necessarily

$C_1 = 0$, $C_r \neq 0$ only for even r, with largest value $r = 2k$

$$\Rightarrow \varepsilon = \underbrace{2k + |m|}_{n} + 1 = n+1 \Rightarrow \boxed{E_n = (n+1)\hbar\omega}$$

E_n is degenerate: For each n, we have the possible combinations:

$n = 2s$: $\quad k = 0, 1, \ldots, s$; $m = \pm(n-2k)$

so two m-values for $k = 0, 1, \ldots, s-1$ $(|m| \neq 0)$,

one m value for $k = s$

$\Rightarrow$ degeneracy $= 2s+1 = \underset{\sim}{n+1}$

$n = 2s+1$: $\quad k = 0, 1, \ldots, s$; $m = \pm(n-2k)$ odd

$\Rightarrow$ degeneracy $= 2(s+1) = (2s+1)+1 = \underset{\sim}{n+1}$

$\Rightarrow$ $\boxed{\text{Each } E_n \text{ is } (n+1)\text{-fold degenerate}}$

- The lowest states:

$\underset{\sim}{n=0}$: $\quad k = m = 0 \qquad \psi_{Em} \equiv \psi_{nm} = \psi_{00} = C_0 e^{-\xi^2/2} = C_0 e^{-\frac{\mu\omega\rho^2}{2\hbar}}$

normalize: $\int \rho\, d\rho\, d\varphi\, |\psi_{00}|^2 = 1 \Rightarrow \psi_{00} = \sqrt{\frac{\mu\omega}{\pi\hbar}}\, e^{-\frac{\mu\omega\rho^2}{2\hbar}}$

$= \psi_0(x)\psi_0(y)$!

$\underset{\sim}{n=1}$: $k=0, \quad m=\pm 1$

$$\psi_{1,\pm 1} = \tilde{c}_0 \, \xi \, e^{-\xi^2/2} \, \frac{e^{\pm i\varphi}}{\sqrt{2\pi}}$$

normalize $\Rightarrow$ $$\boxed{\psi_{1,\pm 1} = \sqrt{\frac{\mu\omega}{\pi\hbar}} \sqrt{\frac{\mu\omega}{\hbar}} \, \rho \, e^{-\frac{\mu\omega\rho^2}{2\hbar}} e^{\pm i\varphi}}$$

12.4 Angular momentum in 3 dimensions

Obvious generalization of what we found in 2d to 3d:

$$\boxed{\hat{\vec{L}} = (\hat{L}_x, \hat{L}_y, \hat{L}_z) = \hat{\vec{X}} \times \hat{\vec{P}}}$$ angular momentum vector operator

$$\hat{L}_x = \hat{Y}\hat{P}_z - \hat{Z}\hat{P}_y$$
$$\hat{L}_y = \hat{Z}\hat{P}_x - \hat{X}\hat{P}_z$$
$$\hat{L}_z = \hat{X}\hat{P}_y - \hat{Y}P_x$$

These satisfy the following commutation relations:

$$[\hat{L}_x, \hat{L}_y] = i\hbar \hat{L}_z, \; [\hat{L}_y, \hat{L}_z] = i\hbar \hat{L}_x, \; [\hat{L}_z, \hat{L}_x] = i\hbar \hat{L}_y$$

or $\boxed{\hat{\vec{L}} \times \hat{\vec{L}} = i\hbar \hat{\vec{L}}}$ or $\boxed{[\hat{L}_i, \hat{L}_j] = i\hbar\, \varepsilon_{ijk} \underbrace{\hat{L}_k}}$ (sum over k implied)

(Proof: e.g. $[\hat{L}_y, \hat{L}_z] = [\hat{Z}\hat{P}_x - \hat{X}\hat{P}_z, \hat{X}\hat{P}_y - \hat{Y}\hat{P}_x] =$

$$= [\hat{Z}\hat{P}_x, \hat{X}\hat{P}_y] - [\hat{X}\hat{P}_z, \hat{X}\hat{P}_y] - [\hat{Z}\hat{P}_x, \hat{Y}\hat{P}_x] + [\hat{X}\hat{P}_z, \hat{Y}\hat{P}_x]$$

$$= \hat{Z}(-i\hbar\hat{\mathbb{1}})\hat{P}_y - 0 - 0 + \hat{P}_z(i\hbar\mathbb{1})Y$$

$$= i\hbar(\hat{P}_z\hat{Y} - \hat{Z}\hat{P}_y) = i\hbar(\hat{Y}\hat{P}_z - \hat{Z}\hat{P}_y) = i\hbar \hat{L}_x \checkmark$$)

Total angular momentum operator

$$\boxed{\hat{\vec{L}}^2 = \hat{L}_x^2 + \hat{L}_y^2 + \hat{L}_z^2}$$

It satisfies $\boxed{[\hat{L}^2, \hat{L}_i] = 0}$ $\forall i = 1,2,3$

Finite rotation operators:

$$\hat{U}[\hat{R}(\theta_k \vec{e}_k)] = e^{-i/\hbar\, \theta_k \hat{L}_k} \qquad k = 1, 2, \text{or } 3$$

rotation by angle θ_k around axis $\vec{e}_k$

Rotation around an arbitrary axis $\hat{\vec{n}}$ by angle θ:

$$\hat{\mathcal{U}}[\hat{R}(\theta\hat{\vec{n}})] \equiv \hat{\mathcal{U}}[\hat{R}(\vec{\vartheta})] \qquad \vec{\vartheta} = \theta\hat{\vec{n}}$$

$$= e^{-\frac{i}{\hbar}\vec{\theta}\cdot\hat{\vec{L}}} \equiv e^{-i/\hbar\,\theta\,\hat{\vec{n}}\cdot\hat{\vec{L}}} = e^{-\frac{i}{\hbar}\theta L_{\hat{\vec{n}}}}$$

Note: We can write $\vec{\theta} = \theta_i \vec{e}_i$, but

$$\hat{\mathcal{U}}[\hat{R}(\vec{\theta})] = \hat{\mathcal{U}}[\hat{R}(\theta_1\vec{e}_1 + \theta\vec{e}_2 + \theta\vec{e}_3)]$$

$$= e^{-\frac{i}{\hbar}(\theta_1\hat{L}_x + \theta_2\hat{L}_y + \theta_3\hat{L}_z)} \qquad \boxed{[L_i, L_j] \neq 0!}$$

$$\neq e^{-i/\hbar\,\theta_1 L_x}\, e^{-i/\hbar\,\theta_2\hat{L}_y}\, e^{-\frac{i}{\hbar}\theta_3\hat{L}_z}$$

$$= \hat{\mathcal{U}}[\hat{R}(\theta_1\vec{e}_x)]\,\hat{\mathcal{U}}[\hat{R}(\theta_2\vec{e}_y)]\,\hat{\mathcal{U}}[\hat{R}(\theta_3\vec{e}_z)]$$

Rotational Invariance:

If the Hamiltonian is invariant under arbitrary rotations,

$$\hat{\mathcal{U}}^{+}[R]\,\hat{H}\,\hat{\mathcal{U}}[\hat{R}] = \hat{H}$$

it is in particular invariant under infinitesimal rotations around each of the x, y, z axes, from which it follows that

$$[\hat{H}, \hat{L}_i] = 0 \quad \text{for } i = 1, 2, \text{and } 3$$

Then

$$[\hat{H}, \hat{\vec{L}}^2] = [\hat{H}, \hat{L}_x^2] + [\hat{H}, \hat{L}_y^2] + [\hat{H}, \hat{L}_z^2] = 0$$

Hence $\vec{L}^2$ and all three components of $\vec{L}$ are

conserved: $\boxed{\frac{d}{dt}\langle\hat{\vec{L}}^2\rangle = \frac{d}{dt}\langle\hat{L}_x\rangle = \frac{d}{dt}\langle\hat{L}_y\rangle = \frac{d}{dt}\langle\hat{L}_z\rangle = 0}$

$\Rightarrow \exists$ common eigenbasis of $\hat{H}, \hat{\vec{L}}^2, \hat{L}_x$ or $\hat{L}_y$ or $\hat{L}_z$, but not of all three $\hat{L}_i$ since they don't commute