The free particle in spherical coordinates:

$$\psi_{E\ell m}(r,\vartheta,\varphi) = R_{E\ell}(r)\, Y_{\ell m}(\vartheta,\varphi) = \frac{u_{E\ell}(r)}{r}\, Y_{\ell m}(\vartheta,\varphi)$$

$$\left(\frac{d^2}{dr^2} + k^2 - \frac{\ell(\ell+1)}{r^2}\right) u_{E\ell}(r) = 0 \qquad \left(k^2 = \frac{2\mu E}{\hbar^2}\right.$$

Use scaled radial coordinate $\rho = kr$:

$$\left(-\frac{d^2}{d\rho^2} + \frac{\ell(\ell+1)}{\rho^2}\right) u_\ell(\rho) = u_\ell(\rho) \qquad \text{(E is hidden in } \rho = kr)$$

Similar form to harmonic oscillator, except we have potential $\frac{1}{\rho^2}$ instead of ρ^2. Define raising and lowering operators in corresponding analogy:

$$\hat{d}_\ell \leftrightarrow d_\ell \equiv \frac{d}{d\rho} + \frac{\ell+1}{\rho} \qquad \hat{d}_\ell^+ \leftrightarrow d_\ell^+ = -\frac{d}{d\rho} + \frac{\ell+1}{\rho}$$

($\frac{d}{d\rho}$ is anti-Hermitean!)

$$\overset{\text{S.Eq.}}{\Rightarrow} \boxed{(\hat{d}_\ell \hat{d}_\ell^+)\, u_\ell = u_\ell}$$

Apply $\hat{d}_\ell$: $\Rightarrow (\hat{d}_\ell^+ \hat{d}_\ell)(\hat{d}_\ell^+ u_\ell) = (\hat{d}_\ell^+ u_\ell)$

$\Rightarrow \hat{d}_\ell^+ u_\ell$ is eigenstate of $\hat{d}_\ell^+ \hat{d}_\ell$ with eigenvalue 1

You can check that:

$$\hat{d}_\ell^+ \hat{d}_\ell = \hat{d}_{\ell+1} \hat{d}_{\ell+1}^+$$

So we have $\left(\hat{d}_{\ell+1}\hat{d}_{\ell+1}^+\right)\left(\hat{d}_\ell^+ u_\ell\right) = \lambda\left(\hat{d}_\ell^+ u_\ell\right)$

i.e. $\hat{d}_\ell^+ u_\ell$ solves the S. Eq. for $\ell+1$:

$$\boxed{\hat{d}_\ell^+ u_\ell = c_\ell\, u_{\ell+1}}$$

For the moment we will set $c_\ell = 1$ since we haven't normalized u_ℓ yet. We see that $\hat{d}_\ell^+$ raises ℓ by 1. Given u_0, we can generate all u_ℓ by repeatedly applying $\hat{d}_\ell$.

Now, for $\ell=0$, the S. Eq. reads

$$\frac{d^2}{d\rho^2}u_0 = -u_0 \quad \Rightarrow \quad \boxed{u_0^A(\rho) = \sin\rho} \quad \left(\underset{\text{irregular} \to \text{throw away}}{u_0^B(\rho) = -\cos\rho}\right)$$

(we need to include this solution in general for solutions in regions that do not include the origin)

Next, construct u_ℓ for $\ell \neq 0$:

$$u_{\ell+1} = \hat{d}_\ell^+ u_\ell \quad \Rightarrow \quad \rho R_{\ell+1} = \hat{d}_\ell^+(\rho R_\ell) = \left(-\frac{d}{d\rho} + \frac{\ell+1}{\rho}\right)(\rho R_\ell)$$

$$\Rightarrow \quad R_{\ell+1} = \left(-\frac{d}{d\rho} + \frac{\ell}{\rho}\right)R_\ell = \rho^\ell\left(-\frac{d}{d\rho}\right)\frac{R_\ell}{\rho^\ell}$$

$$\Rightarrow \quad \frac{R_{\ell+1}}{\rho^{\ell+1}} = \left(-\frac{1}{\rho}\frac{d}{d\rho}\right)\frac{R_\ell}{\rho^\ell} = \left(-\frac{1}{\rho}\frac{d}{d\rho}\right)^2\frac{R_{\ell-1}}{\rho^{\ell-1}} = \dots$$

$$= \left(-\frac{1}{\rho}\frac{d}{d\rho}\right)^{\ell+1}\frac{R_0}{\rho^0} = \left(-\frac{1}{\rho}\frac{d}{d\rho}\right)^{\ell+1} R_0$$

$$\Rightarrow \quad \boxed{R_\ell(\rho) = (-\rho)^\ell \left(\frac{1}{\rho}\frac{\partial}{\partial\rho}\right)^\ell R_0}$$

$$\Rightarrow R_\ell^A \equiv j_\ell = (-\rho)^\ell \left(\frac{1}{\rho}\frac{\partial}{\partial\rho}\right)^\ell \left(\frac{\sin\rho}{\rho}\right) \quad \text{(regular)}$$ spherical Bessel fcts.

$$R_\ell^B \equiv n_\ell = (-\rho)^\ell \left(\frac{1}{\rho}\frac{\partial}{\partial\rho}\right)^\ell \left(-\frac{\cos\rho}{\rho}\right) \quad \text{(irregular)}$$ spherical Neumann functions

See p. 348 for explicit expressions for $j_0, j_1, j_2, n_0, n_1, n_2$

Asymptotic behavior at $\rho \to \infty$:

$$j_\ell(\rho) \xrightarrow[\rho\to\infty]{} \frac{1}{\rho}\sin\left(\rho - \frac{\ell\pi}{2}\right)$$

$$n_\ell(\rho) \xrightarrow[\rho\to\infty]{} -\frac{1}{\rho}\cos\left(\rho - \frac{\ell\pi}{2}\right)$$

Asymptotics at $\rho \to 0$:

$$j_\ell(\rho) \xrightarrow[\rho\to 0]{} \frac{\rho^\ell}{(2\ell+1)!!}$$

$$n_\ell(\rho) \xrightarrow[\rho\to 0]{} -\frac{(2\ell-1)!!}{\rho^{\ell+1}}$$

The full eigenstate solution for a free particle is then

$$\psi_{E\ell m}(r,\vartheta,\varphi) = \langle r\vartheta\varphi | E\ell m\rangle = j_\ell(kr)\, Y_{\ell m}(\vartheta,\varphi), \qquad E = \frac{\hbar^2 k^2}{2\mu}$$

with normalization

$$\int r^2 dr\, d\Omega\, \psi^*_{E\ell m}(r,\vartheta,\varphi)\, \psi_{E'\ell' m'}(r,\vartheta,\varphi) = \frac{\pi}{2k^2}\,\delta(k-k')\,\delta_{\ell\ell'}\,\delta_{mm'}$$

$$\left(\int_0^\infty j_\ell(kr)\, j_\ell(k'r)\, r^2 dr = \frac{\pi}{2k^2}\,\delta(k-k') \right)$$

Connection to Cartesian coordinates:

$$\psi_E(x,y,z) = \frac{1}{(2\pi\hbar)^{3/2}} e^{i\vec{p}\cdot\vec{r}/\hbar} \qquad E = \frac{p^2}{2\mu} = \frac{\hbar^2 k^2}{2\mu}$$

$$\Rightarrow \psi_E(r,\vartheta,\varphi) = \frac{e^{ikr\cos\vartheta}}{(2\pi\hbar)^{3/2}}$$

We can expand this into the complete set of spherical eigenfunctions:

$$e^{ikr\cos\vartheta} = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} C_{lm}\, j_l(kr)\, Y_{lm}(\vartheta,\varphi)$$

The left hand side is independent of φ, so $C_{lm} = 0 \ \forall m \neq 0$.

Further $Y_{l0}(\vartheta,\varphi) = \sqrt{\frac{2l+1}{4\pi}}\, P_l(\cos\vartheta)$.

$$\Rightarrow e^{ikr\cos\vartheta} = \sum_{l=0}^{\infty} C_l\, j_l(kr)\, P_l(\cos\vartheta) \qquad C_l \equiv C_{l0}\sqrt{\frac{2l+1}{4\pi}}$$

Using the orthogonality relation

$$\int_{-1}^{1} dx\, P_l(x)\, P_{l'}(x) = \frac{2}{2l+1}\,\delta_{ll'}$$

one can show that

$$C_l = i^l(2l+1)$$

$$\Rightarrow \boxed{e^{ikr\cos\vartheta} = \sum_{l=0}^{\infty} i^l (2l+1)\, j_l(kr)\, P_l(\cos\vartheta)}$$

The isotropic oscillator

$$\hat{H} = \frac{\hat{\vec{P}}^2}{2\mu} + \frac{\mu\omega^2}{2}\hat{\vec{R}}^2 \quad , \quad \psi_{E\ell m} = \frac{u_{E\ell}(r)}{r} Y_{\ell m}(\vartheta,\varphi)$$

$$\Rightarrow \left[\frac{d^2}{dr^2} + \frac{2\mu}{\hbar^2}\left(E - \frac{\mu\omega^2 r^2}{2} - \frac{\ell(\ell+1)\hbar^2}{2\mu r^2}\right)\right] u_{E\ell} = 0$$

Use scaled coordinates $\xi = \frac{r}{r_0}$, $r_0 = \sqrt{\frac{\hbar}{\mu\omega}}$

(as in the 1-d case). Rewrite the S. Eq. in ξ.

Study limit $\xi \to \infty$ and find

$$\lim_{\xi\to\infty} u(\xi) \sim e^{-\xi^2/2}$$

(again like in the 1-d case, since for $\xi \to \infty$ the centrifugal barrier can be neglected)

So we write $u(\xi) = v(\xi)\, e^{-\xi^2/2}$

and find that v must satisfy

$$\boxed{v'' - 2\xi v' + \left(2\varepsilon - 1 - \frac{\ell(\ell+1)}{\xi^2}\right) v = 0 \qquad \text{where} \quad \varepsilon = \frac{E}{\hbar\omega}}$$

(Similar to the 1-d case, except for the centrifugal term.)
Eq. (7.3.11)

Using the known behavior $u(\xi) \sim \xi^{\ell+1}$ near the origin,

we write the ansatz

$$v(\xi) = \xi^{\ell+1} \sum_{n=0}^{\infty} c_n \xi^n$$

and derive a recursion relation for c_n.

The recursion relation breaks off (this is needed for normalizability) if and only if

$$\varepsilon = \left(2k+\ell+\tfrac{3}{2}\right) \Rightarrow E_{k\ell} = \left(2k+\ell+\tfrac{3}{2}\right)\hbar\omega \quad (k=0,1,2,\dots)$$

We define the <u>principal quantum number</u>

$$n = 2k+\ell$$

$$\Rightarrow \boxed{E_n = \left(n+\tfrac{3}{2}\right)\hbar\omega} \qquad (n=0,1,2,\dots)$$

For each n, we have several allowed ℓ values (degeneracy!):

$$\ell = n-2k \rightarrow \ell = n, n-2, \dots, 1 \text{ or } 0.$$

$n=0$:	$\ell=0$	$m=0$
$n=1$:	$\ell=1$	$m=\pm1, 0$.
$n=2$:	$\ell=2, 0$	$m=\pm2, \pm1, 0$; $m=0$
$n=3$:	$\ell=3, 1$	$m=\pm3, \pm2, \pm1, 0$; $m=\pm1, 0$

...

The $(2\ell+1)$-fold degeneracy in m is due to rotational invariance. The degeneracy between states of different ℓ and same n is mysterious $\rightarrow$ "accidental degeneracy". It is, however, not accidental at all: $\hat{H}$ has additional symmetries, see Chapter 15.4.