The free particle in spherical coordinates:

\[ \psi_{\text{free}}(r, \theta, \phi) = R_{\text{free}}(r) Y_{\text{free}}(\theta, \phi) = \frac{U_{\text{free}}(r)}{r} Y_{\text{free}}(\theta, \phi) \]

\[ \left( \frac{d^2}{dr^2} + \frac{l(l+1)}{r^2} \right) U_{\text{free}}(r) = 0 \quad (k^2 = \frac{2mE}{\hbar^2}) \]

Use scaled radial coordinate \( \rho = kr \):

\[ \left( -\frac{d^2}{d\rho^2} + \frac{l(l+1)}{\rho^2} \right) U_\rho(\rho) = U_\rho(\rho) \quad (E \text{ is hidden in } \rho = kr) \]

Similar form to harmonic oscillator, except we have potential \( \frac{1}{\rho^2} \) instead of \( \rho^2 \). Define raising and lowering operators in corresponding analogy:

\[ \hat{d}_- \leftrightarrow \hat{d}_+ = \frac{1}{\rho} \frac{d}{d\rho} + \frac{l+1}{\rho} \]

\[ \hat{d}_- \leftrightarrow \hat{d}^+ = -\frac{1}{\rho} \frac{d}{d\rho} + \frac{l+1}{\rho} \]

\[ (\frac{d}{d\rho} \text{ is anti-Hermitian}) \]

S.Eq. \[ \Rightarrow \quad (\hat{d}_- \hat{d}^+ \psi) = \psi \]

Apply \( \hat{d}_- \):

\[ \Rightarrow (\hat{d}_+ \hat{d}_-) (\hat{d}^+ \psi) = (\hat{d}^+ \psi) \]

\[ \Rightarrow \hat{d}^+ \psi \text{ is eigenstate of } \hat{d}_- \text{ with eigenvalue} \]

\[ \text{eigenvalue!} \]
You can check that
\[ \hat{\Delta}_x^+ \hat{\Delta}_x = \hat{\Delta}_{x+1}^+ \hat{\Delta}_{x+1} \]
so we have \((\hat{\Delta}_x^+ \hat{\Delta}_x^-)(\hat{\Delta}_x^+ U_e) = \hat{\Delta}_x^+ U_e)\)
i.e. \(\hat{\Delta}_x^+ U_e\) solves the S.Eq. for \(l+1\):
\[ \hat{\Delta}_x^+ U_e = C_e U_{x+1} \]
For the moment we will set \(C_e=1\) since we haven't normalized \(U_e\) yet. We see that \(\hat{\Delta}_x^+\) raises \(l\) by 1. Given \(U_0\), we can generate all \(U_e\) by repeatedly applying \(\hat{\Delta}_x^+\).

Now, for \(l=0\), the S.Eq. reads
\[ \frac{d^2}{d\rho^2} U_0 = -U_0 \Rightarrow U_0^\Lambda(\rho) = \sin \rho \]
\[ U_0^B(\rho) = -\cos \rho \]
irregular \(\Rightarrow\) throw away
(we need to include this solution in general for solutions in regions that do not include the origin)

Next, construct \(U_x\) for \(l=0\):
\[ U_{x+1} = \hat{\Delta}_x^+ U_x \Rightarrow \phi \text{Re}_{x+1} = \frac{1}{\rho} \left( \phi \text{Re} \right) = \left( -\frac{1}{\rho} \frac{d}{d\phi} \right) \left( \frac{d}{d\phi} \right) \left( \frac{d}{d\phi} \right) \]
\[ \Rightarrow \text{Re}_{x+1} = \left( -\frac{1}{\rho} \frac{d}{d\phi} + \frac{\rho}{\phi} \right) \text{Re} = \rho \left( -\frac{1}{\rho} \frac{d}{d\phi} \right) \frac{\text{Re}}{\rho^2} \]
\[ \Rightarrow \frac{\text{Re}_{x+1}}{\rho^{x+1}} = \left( -\frac{1}{\rho} \frac{d}{d\phi} \right) \frac{\text{Re}}{\rho^2} = \left( -\frac{1}{\rho} \frac{d}{d\phi} \right)^2 \frac{\text{Re}_{x+1}}{\rho^{x+1}} = \ldots \]
\[ = \left( -\frac{1}{\rho} \frac{d}{d\phi} \right)^{x+1} \frac{R_0}{\rho^x} = \left( -\frac{1}{\rho} \frac{d}{d\phi} \right)^{x+1} \frac{R_0}{\rho^x} \]
\[
R_{e}(\rho) = (-\rho)^{l} \left( \frac{1}{\rho} \frac{\partial}{\partial \rho} \right)^{l} R_{0}
\]

\[
\Rightarrow R_{e}^{A} = j_{e} = (-\rho)^{l} \left( \frac{1}{\rho} \frac{\partial}{\partial \rho} \right)^{l} \left( \frac{\sin \rho}{\rho} \right) \quad \text{(regular)} \quad \text{spherical Bessel fets.}
\]

\[
R_{e}^{B} = n_{e} = (-\rho)^{l} \left( \frac{1}{\rho} \frac{\partial}{\partial \rho} \right)^{l} \left( -\cos \rho \right) \quad \text{(irregular)} \quad \text{spherical Neumann functions}
\]

See p. 348 for explicit expressions for \( j_{0}, j_{1}, j_{2}, n_{0}, n_{1}, n_{2} \)

Asymptotic behavior at \( \rho \to \infty \):

\[
\lim_{\rho \to \infty} j_{e}(\rho) = \frac{1}{\rho} \sin(\rho - \frac{\rho}{2})
\]

\[
\lim_{\rho \to \infty} n_{e}(\rho) = -\frac{1}{\rho} \cos(\rho - \frac{\rho}{2})
\]

Asymptotics at \( \rho \to 0 \):

\[
\lim_{\rho \to 0} j_{e}(\rho) = \frac{0}{(2l+1)!!}
\]

\[
\lim_{\rho \to 0} n_{e}(\rho) = -\frac{(2l-1)!!}{\rho^{l+1}}
\]

The full eigenstate solution for a free particle is then

\[
\Psi_{E \ell m}(r, \theta, \phi) = \langle r d \theta d \phi | E \ell m \rangle = j_{\ell}(kr) Y_{\ell m}(\theta, \phi), \quad E = \frac{\hbar^{2} k^{2}}{2m}
\]

with normalization

\[
\int r^{2} dr d\Omega \Psi_{E \ell m}^{*}(r, \theta, \phi) \Psi_{E \ell' m'}(r, \theta, \phi) = \frac{\pi}{2k} \delta(k - k') \delta_{\ell \ell'} \delta_{m m'}
\]

\[
\left( \int j_{\ell}(kr) j_{\ell}(kr) r^{2} dr = \frac{\pi}{2k^{2}} \delta(k - k') \right)
\]
Connection to Cartesian coordinates,

\[ \psi_E(x,y,z) = \frac{1}{(2\pi)^{\frac{3}{2}}} e^{i\mathbf{p} \cdot \mathbf{r}/\hbar} \]

\[ E = \frac{p^2}{2m} = \frac{\hbar^2 k^2}{2m} \]

\[ \Rightarrow \psi_E(r,\theta,\phi) = \frac{e^{ikr \cos \theta}}{(2\pi \hbar)^{\frac{3}{2}}} \]

We can expand this into the complete set of spherical eigenfunctions:

\[ e^{ikr \cos \theta} = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} C_{lm} j_l(kr) Y_{lm}(\theta,\phi) \]

The left hand side is independent of \( \phi \), so \( C_{lm} \neq 0 \) for \( m \neq 0 \).

Further \( Y_{l0}(\theta,\phi) = \sqrt{\frac{2l+1}{4\pi}} P_l(\cos \theta) \).

\[ \Rightarrow e^{ikr \cos \theta} = \sum_{l=0}^{\infty} C_l j_l(kr) P_l(\cos \theta) \quad C_l = C_{l0} \sqrt{\frac{2l+1}{4\pi}} \]

Using the orthogonality relation

\[ \int_{-1}^{1} P_l(x) P_{l'}(x) \, dx = \frac{2}{2l+1} \delta_{ll'} \]

one can show that

\[ C_l = i^l (2l+1) \]

\[ \Rightarrow \left( e^{ikr \cos \theta} = \sum_{l=0}^{\infty} i^l (2l+1) j_l(kr) P_l(\cos \theta) \right) \]
The isotropic oscillator

\[ H = \frac{p^2}{2\mu} + \frac{\mu \omega^2 r^2}{2} \quad \text{and} \quad \Psi_{E_n} = \frac{U_{E_n}(r)}{r} \quad Y_{lm}(\theta, \phi) \]

\[ \Rightarrow \left[ \frac{d^2}{dr^2} + \frac{2\mu}{\hbar^2} \left( E - \frac{\mu \omega^2 r^2}{2} - \frac{\hbar (l+1) \omega}{2} \right) \right] U_{E_n} = 0 \]

Use scaled coordinates \( \bar{r} = \frac{r}{r_0}, \quad r_0 = \sqrt{\frac{\hbar}{\mu \omega}} \)

(as in the 1-d case). Rewrite the S. Eq. in \( \bar{r} \).

Study limit \( \bar{r} \to \infty \) and find

\[ \lim_{\bar{r} \to \infty} U(\bar{r}) \sim e^{-3/2} \]

(again like in the 1-d case, since for \( \bar{r} \to \infty \)
the centrifugal barrier can be neglected.)

So we write \( U(\bar{r}) = v(\bar{r}) e^{-3/2} \)

and find that \( v \) must satisfy

\[ v'' - 2\bar{r} v' + \left( 2\bar{r} - \frac{l(l+1)}{\bar{r}^2} \right) v = 0 \quad \text{where} \quad \bar{r} = \frac{\bar{r}}{r_0} \]

(Similar to the 1-d case, except for the centrifugal term.)

Eq. (7.3.11)

Using the known behavior \( U(\bar{r}) \sim \bar{r}^{l+1} \) near the origin,

we write the ansatz

\[ U(\bar{r}) \sim \bar{r}^{l+1} \sum_{n=0}^{\infty} C_n \bar{r}^{\gamma_n} \]

and derive a recursion relation for \( C_n \).
The recursion relation breaks off (this is needed for normalizability) if and only if

\[ \varepsilon = (2k+l+\frac{3}{2}) \Rightarrow E_{n,k} = (2k+l+\frac{3}{2}) \hbar \omega \quad (k=0,1,2,\ldots) \]

We define the principal quantum number

\[ n = 2k+l \]

\[ \Rightarrow E_n = (n+\frac{3}{2}) \hbar \omega \quad (n=0,1,2,\ldots) \]

For each \( n \), we have several allowed \( l \) values (degeneracy!):

\[ l = n - 2k \rightarrow l = n, n-2, \ldots, 0, \text{or} 0. \]

\[ \begin{align*}
  n=0 : & \quad l=0 \quad m=0 \\
  n=1 : & \quad l=1 \quad m=\pm 1, 0 \\
  n=2 : & \quad l=2, 0 \quad m=\pm 2, \pm 1, 0 \quad 0 \quad m=0 \\
  n=3 : & \quad l=3, 1 \quad m=\pm 3, \pm 2, \pm 1, 0 \quad 0 \quad m=\pm 1, 0 \\
  \ldots
\end{align*} \]

The \( (2l+1) \)-fold degeneracy in \( m \) is due to rotational invariance. The degeneracy between states of different \( l \) and same \( n \) is mysterious — "accidental degeneracy". It is, however, not accidental at all: it has additional symmetries, see Chapter 15.4.