Physics 7502: Homework Set No. 2

Due date: Tuesday, Jan. 26, 2016, 5:00pm in PRB M2039 (Bowen Shi's office)

Total point value of set: 100 points

Note: All cross products in this problem set must be worked out with the help of the Levi-Civitta tensor ε_{ijk} (ε_{ijk} is totally antisymetric, i.e. it changes sign under interchange of any two indices, and $\varepsilon_{123} = 1$)! Remember $\sum_{n=1}^{3} \varepsilon_{ijn}\varepsilon_{nk\ell} = \delta_{ik}\delta_{j\ell} - \delta_{i\ell}\delta_{jk}$.

Problem 1 (10 pts.): Show that the definition of the orbital angular momentum operator $\hat{\boldsymbol{L}}$ is unambiguous, i.e. that you obtain the same operator (with identical effects on any state $|\psi\rangle$) if you start from the classical expressions $\boldsymbol{L} = -\boldsymbol{p} \times \boldsymbol{x}$ and $\boldsymbol{L} = \boldsymbol{x} \times \boldsymbol{p}$ and then substitute $\boldsymbol{x} \mapsto \hat{\boldsymbol{X}}$, $\boldsymbol{p} \mapsto \hat{\boldsymbol{P}}$. (Note the different order of multiplication and differentiation that you obtain in the two cases!)

Problem 2 (30 pts.): For *any* vector operator $\hat{\boldsymbol{v}}$ constructed from $\hat{\boldsymbol{X}}$ and $\hat{\boldsymbol{P}}$, prove that $[\hat{L}_i, \hat{v}_j] = i\hbar \sum_{k=1}^{3} \varepsilon_{ijk}\hat{v}_k$.
(Hint: Write $\boldsymbol{v} = a(\boldsymbol{x}^2, \boldsymbol{x}\cdot\boldsymbol{p}, \boldsymbol{p}^2)\, \boldsymbol{x} + b(\boldsymbol{x}^2, \boldsymbol{x}\cdot\boldsymbol{p}, \boldsymbol{p}^2)\, \boldsymbol{p} + c(\boldsymbol{x}^2, \boldsymbol{x}\cdot\boldsymbol{p}, \boldsymbol{p}^2)\, \boldsymbol{x}\times\boldsymbol{p}$ (where a, b, c are arbitrary functions). Derive commutation relations between the components of $\hat{\boldsymbol{L}}$ and those of $\hat{\boldsymbol{X}}$ and $\hat{\boldsymbol{P}}$ and show that $\hat{\boldsymbol{L}}$ commutes with the scalar operators $\hat{a}$, $\hat{b}$, $\hat{c}$. Proceed to show that $[\hat{L}_i, \hat{v}_j] = i\hbar\varepsilon_{ijk}\hat{v}_k$ (sum over k implied).)

Problem 3 (10 pts.): Prove the identity $\hat{\boldsymbol{L}} \times \hat{\boldsymbol{L}} = i\hbar\hat{\boldsymbol{L}}$ for the orbital angular momentum operator $\hat{\boldsymbol{L}} = \hat{\boldsymbol{X}} \times \hat{\boldsymbol{P}}$.

Problem 4 (20 pts.): From the definition $\hat{\boldsymbol{L}} = \hat{\boldsymbol{X}} \times \hat{\boldsymbol{P}}$, derive the following expressions in the position representation $\hat{\boldsymbol{P}} \to -i\hbar\boldsymbol{\nabla}$ in spherical coordinates:

$$
\begin{aligned}
\hat{L}_x &\to L_1 = i\hbar\left(\sin\phi\,\partial_\theta + \cot\theta\cos\phi\,\partial_\phi\right), \\
\hat{L}_y &\to L_2 = i\hbar\left(-\cos\phi\,\partial_\theta + \cot\theta\sin\phi\,\partial_\phi\right), \\
\hat{L}_z &\to L_3 = -i\hbar\,\partial_\phi, \\
\hat{\boldsymbol{L}}^2 &\to \boldsymbol{L}^2 = -\hbar^2\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial^2\phi}\right].
\end{aligned}
$$

Hint: It's easiest to prove the first three backwards. For the last, express $\boldsymbol{L}^2$ through $L_\pm = L_x \pm iL_y$ to simplify the algebra.

Problem 5 (10 pts.): Exercise 12.5.1 (Shankar, p. 325)

Problem 6 (10 pts.): Exercise 12.5.3 (Shankar, p. 329)

Problem 7 (10 pts.): Exercise 12.5.5, part (1) only (Shankar, p. 332)