Physics 7502: Homework Set No. 2

Due date: Tuesday, Jan. 26, 2016, 5:00pm
in PRB M2039 (Bowen Shi’s office)

Total point value of set: 100 points

Note: All cross products in this problem set must be worked out with the help of the Levi-Civitta tensor $\varepsilon_{ijk}$ ($\varepsilon_{ijk}$ is totally antisymetric, i.e. it changes sign under interchange of any two indices, and $\varepsilon_{123} = 1$)! Remember $\sum_{n=1}^{3} \varepsilon_{ijn} \varepsilon_{nk\ell} = \delta_{ik} \delta_{j\ell} - \delta_{i\ell} \delta_{jk}$.

Problem 1 (10 pts.): Show that the definition of the orbital angular momentum operator $\hat{L}$ is unambiguous, i.e. that you obtain the same operator (with identical effects on any state $|\psi\rangle$) if you start from the classical expressions $L = -p \times x$ and $L = x \times p$ and then substitute $x \mapsto \hat{X}$, $p \mapsto \hat{P}$. (Note the different order of multiplication and differentiation that you obtain in the two cases!)

Problem 2 (30 pts.): For any vector operator $\hat{v}$ constructed from $\hat{X}$ and $\hat{P}$, prove that $[\hat{L}_i, \hat{v}_j] = i\hbar \varepsilon_{ijk} \hat{v}_k$. (Hint: Write $v = a(x^2, x \cdot p, p^2) x + b(x^2, x \cdot p, p^2) p + c(x^2, x \cdot p, p^2) x \times p$ (where $a, b, c$ are arbitrary functions). Derive commutation relations between the components of $\hat{L}$ and those of $\hat{X}$ and $\hat{P}$ and show that $\hat{L}$ commutes with the scalar operators $\hat{a}$, $\hat{b}$, $\hat{c}$. Proceed to show that $[\hat{L}_i, \hat{v}_j] = i\hbar \varepsilon_{ijk} \hat{v}_k$ (sum over $k$ implied).)

Problem 3 (10 pts.): Prove the identity $\hat{L} \times \hat{L} = i\hbar \hat{L}$ for the orbital angular momentum operator $\hat{L} = \hat{X} \times \hat{P}$.

Problem 4 (20 pts.): From the definition $\hat{L} = \hat{X} \times \hat{P}$, derive the following expressions in the position representation $\hat{P} \mapsto -i\hbar \nabla$ in spherical coordinates:

\[
\begin{align*}
\hat{L}_x &\rightarrow L_1 = i\hbar (\sin \phi \partial_\theta + \cot \phi \cos \phi \partial_\phi), \\
\hat{L}_y &\rightarrow L_2 = i\hbar (-\cos \phi \partial_\theta + \cot \phi \sin \phi \partial_\phi), \\
\hat{L}_z &\rightarrow L_3 = -i\hbar \partial_\phi, \\
\hat{L}^2 &\rightarrow L^2 = -\hbar^2 \left[ \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial}{\partial \theta} \right) + \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial^2 \phi} \right].
\end{align*}
\]

Hint: It’s easiest to prove the first three backwards. For the last, express $L^2$ through $L_\pm = L_x \pm iL_y$ to simplify the algebra.

Problem 5 (10 pts.): Exercise 12.5.1 (Shankar, p. 325)

Problem 6 (10 pts.): Exercise 12.5.3 (Shankar, p. 329)

Problem 7 (10 pts.): Exercise 12.5.5, part (1) only (Shankar, p. 332)