Example

Suppose that a person plays a game in which his score must be one of the 50 numbers $1, 2, \ldots, 50$ and that each of these 50 numbers is equally likely to be his score. The first time he plays the game, his score is X. He then continues to play the game until he obtains another score Y such that $Y \geq X$. We will assume that, conditional on previous plays, the 50 scores remain equally likely on all subsequent plays. Determine the probability of the event A that $Y = 50$.

$A = [Y = 50]$ $\quad$ $Pr(A) = ?$

Y depends on X

$B_i = [X = i]$ $\quad$ $i = 1, 2, \ldots, 50$

↳ form a partition

use the law of total probability

$$\Pr(A) = \Pr(B_1) \cdot \Pr(A \mid B_1) + \Pr(B_2)\Pr(A \mid B_2) + \ldots + \Pr(B_{50})\Pr(A \mid B_{50})$$

$$\Pr(B_1) = \frac{1}{50}$$

$$\Pr(B_i) = \frac{1}{50} \qquad i = 1, 2, \ldots, 50$$

$$\Pr(A \mid B_1) = \frac{1}{50}$$

$$\Pr(A \mid B_2) = \frac{1}{49}$$

$\Pr(A \mid B_i) =$ $[X = i \Rightarrow Y$ can be any of $i, i+1, \ldots, 50]$

$$= \frac{1}{50 - (i-1)}$$

More facts

- If $A \subset B$ with $\Pr(B) > 0$, what is $\Pr(A \mid B)$?

$$\Pr(A|B) = \frac{\Pr(A \cap B)}{\Pr(B)} = \frac{\Pr(A)}{\Pr(B)}$$

- if A and B are disjoint, with $\Pr(B) > 0$, what is $\Pr(A \mid B)$?

$$\Pr(A|B) = 0 = \frac{\Pr(A \cap B)}{\Pr(B)} = 0$$

- For any two events A, B with $\Pr(B) > 0$

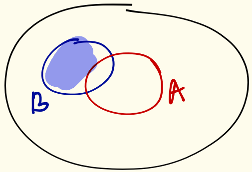
$$\Pr(A^c \mid B) = 1 - \Pr(A \mid B)$$

- For any three events A, B, D such that $\Pr(D) > 0$

$$\Pr(A \cup B \mid D) = \Pr(A \mid D) + \Pr(B \mid D) - \Pr(A \cap B \mid D)$$

exercise

$$\Pr(A^c \mid B) \stackrel{?}{=} 1 - \Pr(A \mid B)$$



$$\Pr(A^c \mid B) = \frac{\Pr(A^c \cap B)}{\Pr(B)} = \frac{\Pr(B \setminus A)}{\Pr(B)} =$$

$$= \frac{\Pr(B \setminus (A \cap B))}{\Pr(B)} = \frac{\Pr(B) - \Pr(A \cap B)}{\Pr(B)}$$

$$= 1 - \frac{\Pr(A \cap B)}{\Pr(B)} = 1 - \Pr(A \mid B)$$

Bayes Theorem

Suppose that you are walking down the street and notice that the Department of Public Health is giving a free medical test for a certain disease. The test is 90 percent reliable in the following sense: If a person has the disease, there is a probability of 0.9 that the test will give a positive response; whereas, if a person does not have the disease, there is a probability of only 0.1 that the test will give a positive response. Data indicate that your chances of having the disease are only 1 in 10,000. However, since the test costs you nothing, and is fast and harmless, you decide to stop and take the test. A few days later you learn that you had a positive response to the test. Now, what is the probability that you have the disease?

$A = [\text{test is positive}]$

$B_1 = [\text{person has disease}]$ $\quad \Pr(B_1) = 1/10000$

$B_2 = [\text{person does not have disease}]$ $\quad \Pr(B_2) = 1 - 1/10000$

$\Pr(A \mid B_1) = .9$ $\quad \Pr(A \mid B_2) = .1$ $\quad \Pr(B_1 \mid A) = ?$

$Pr(B_1)$

$Pr(B_2)$

$Pr(A \mid B_1)$

$Pr(A \mid B_2)$

$$Pr(B_1 \mid A) = \frac{Pr(A \cap B_1)}{Pr(A)} = \frac{Pr(B_1) \cdot Pr(A \mid B_1)}{Pr(A)}$$

$$Pr(A) = Pr(B_1) \cdot Pr(A \mid B_1) + Pr(B_2) \cdot Pr(A \mid B_2)$$

$$\boxed{Pr(B_1 \mid A) = \frac{Pr(B_1) \cdot Pr(A \mid B_1)}{Pr(B_1)\, Pr(A \mid B_1) + Pr(B_2)\, Pr(A \mid B_2)}}$$

Bayes Theorem

Let the events $B_1, \ldots, B_k$ form a partition of the sample space S, such that $\Pr(B_j) > 0$, for $j = 1, 2, \ldots, k$. Let A be another event such that $\Pr(A) > 0$. Then, for every $j = 1, 2, \ldots, k$

$$\Pr(B_j \mid A) = \frac{\Pr(B_j)\Pr(A \mid B_j)}{\sum_{i=1}^{k} \Pr(B_i)\Pr(A \mid B_i)}$$

Three prisoners problem

Three prisoners, A, B and C, are in separate cells and sentenced to death. The governor has selected one of them at random to be pardoned. The warden knows which one is pardoned, but is not allowed to tell. Prisoner A begs the warden to let him know the identity of one of the others who is going to be executed. "If B is to be pardoned, give me C's name. If C is to be pardoned, give me B's name. And if I'm to be pardoned, flip a coin to decide whether to name B or C."
The warden tells A that B is to be executed.

- Prisoner A is pleased because he believes that his probability of surviving has gone up from $1/3$ to $1/2$, as it is now between him and C.
- Prisoner A secretly tells C the news, who is also pleased, because he reasons that A still has a chance of $1/3$ to be the pardoned one, but his chance has gone up to $2/3$.

What is the correct answer?

$A = [\text{A will be pardoned}]$ $\qquad Pr(A) = 1/3$

$B = [\text{B will be pardoned}]$ $\qquad Pr(B) = 1/3$

$C = [\text{C will be pardoned}]$ $\qquad Pr(C) = 1/3$

$W = [\text{B will be executed}] \equiv [\text{info given by warden}]$.

$$Pr(A \mid W) = \frac{Pr(A) \cdot Pr(W \mid A)}{Pr(A) \cdot Pr(W \mid A) + Pr(B)\, Pr(W \mid B) + Pr(C)\, Pr(W \mid C)}$$

$$= \frac{1/3 \cdot 1/2}{1/3 \cdot 1/2 + 1/3 \cdot 0 + \frac{1}{3} \cdot 1} = \frac{1/3 \cdot 1/2}{\frac{1}{3} \cdot \frac{3}{2}} = \frac{1}{3}$$

$Pr(C \mid W) = \frac{2}{3}$

$Pr(B \mid W) = 0$

The Monty Hall problem

Suppose you're on a game show, and you're given the choice of three doors: Behind one door is a car; behind the others, goats. You pick a door, say No. 1, and the host, who knows what's behind the doors, opens another door, say No. 3, which has a goat. He then says to you, "Do you want to pick door No. 2?"

Is it to your advantage to switch your choice?

exercise

Independent events

Definition

Two events A and B are independent if

$$\Pr(A \cap B) = \Pr(A)\Pr(B)$$

Example

Suppose that a balanced die is rolled. Let A be the event that an even number is obtained, and let B be the event that one of the numbers $1, 2, 3$, or 4 is obtained. Show that the events A and B are independent.

$\Pr(A) = \frac{1}{2}$ $\quad \Pr(B) = \frac{4}{6}$ $\quad \Pr(A \cap B) = \frac{2}{6}$

$\frac{1}{2} \cdot \frac{4}{6} = \frac{2}{6}$

A, B – independent.

Example

Suppose that two machines 1 and 2 in a factory are operated independently of each other. Let A be the event that machine 1 will become inoperative during a given 8-hour period, let B be the event that machine 2 will become inoperative during the same period, and suppose that $\Pr(A) = 1/3$ and $\Pr(B) = 1/4$. We shall determine the probability that at least one of the machines will become inoperative during the given period.

$$\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$$

$$= \Pr(A) + \Pr(B) - \Pr(A) \cdot \Pr(B)$$

$$= \frac{1}{3} + \frac{1}{4} - \frac{1}{3} \cdot \frac{1}{4}$$

Independence of complements

If A and B are independent then

- A^c and B^c are independent; exercise
- A^c and B are independent;
- $\underline{A \text{ and } B^c}$ are independent.

$$\Pr(A \cap B^c) = \Pr(A \setminus B) = \Pr(A \setminus (A \cap B)) =$$

$$= \Pr(A) - \Pr(A \cap B) = \Pr(A) - \Pr(A)\Pr(B) =$$

$$= \Pr(A)\left[1 - \Pr(B)\right] =$$

$$= \Pr(A) \cdot \Pr(B^c)$$

Independence of three (or more) events

Events A B and C are independent if

- $\Pr(A \cap B) = \Pr(A)\Pr(B)$
- $\Pr(A \cap C) = \Pr(A)\Pr(C)$
- $\Pr(B \cap C) = \Pr(B)\Pr(C)$
- $\Pr(A \cap B \cap C) = \Pr(A)\Pr(B)\Pr(C)$

Exercise:
Events A, B, C, D are independent if ...

Example

Suppose that a machine produces a defective item with probability p ($0 < p < 1$) and produces a nondefective item with probability $1 - p$. Suppose further that six items produced by the machine are selected at random and inspected, and that the results (defective or nondefective) for these six items are independent. Determine the probability that exactly two of the six items are defective.

$$\binom{6}{2} p^2 (1-p)^4$$

Determine the probability that at least one of the six items in the sample will be defective.

$$1 - (1-p)^6$$

Independence and conditional probability

Events A and B with $\Pr(B) > 0$ are independent if

$$\Pr(A \mid B) = \Pr(A)$$

$$\Pr(A \mid B) = \frac{\Pr(A \cap B)}{\Pr(B)} = \frac{\Pr(A)\,\cancel{\Pr(B)}}{\cancel{\Pr(B)}} = \Pr(A)$$

Mutually independent vs. Mutually exclusive

If events A and B are mutually exclusive, are they independent ? NO

Example

Two students A and B are both registered for a certain course. Assume that student A attends class 80 percent of the time, student B attends class 60 percent of the time, and the absences of the two students are independent.

1. What is the probability that at least one of the two students will be in class on a given day?

   exercise

2. If at least one of the two students is in class on a given day, what is the probability that A is in class that day?

   exercise