Exercise

Suppose that a random variable X has the Binomial distribution with parameters $n = 8$ and $p = 0.7$. Find $\Pr(X \geq 5)$.

$$X \in \{0, 1, 2, \ldots, 8\}$$

$$Pr(X \geq 5) = Pr(X=5) + Pr(X=6) + Pr(X=7) + Pr(X=8)$$

$$Pr(X=i) = \binom{8}{i} .7^{i} .3^{8-i}$$

Uniform distribution

Say X is a discrete random variable such that

$$X \in \{x_1, x_2, \ldots, x_n\}$$

We say that X has a Uniform distribution if the pmf is given by

$$f(x_i) = \Pr(X = x_i) = \frac{1}{n}$$

Suppose that a random variable X has the uniform distribution on the integers $10, \ldots, 20$. Find $\Pr(X = 17)$.

$$\Pr(X = 17) = \frac{1}{11}$$

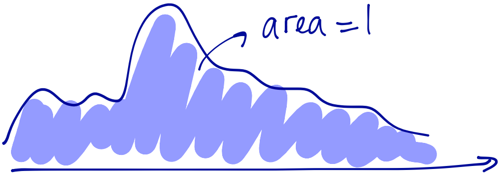
3.2 Continuous Random Variabls and Their Distributions

Recall: A continuous random variable takes values in a subinterval of the real line

$$X \in [a, b] \quad \text{or} \quad X \in (a, \infty) \quad \text{or} \quad X \in \mathbb{R}$$

In this case, the distribution of a random variable is specified through a **probability density function** (**pdf**) $f(x)$ which has the following two properties

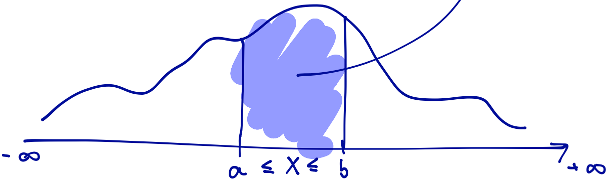
1. $f(x) \geq 0$
2. $\int_{-\infty}^{\infty} f(x)\, dx = 1$



Calculating probabilities with continuous random variables

Say X is a continuous random variable, with **probability density function** (**pdf**) $f(x)$. The probability that X takes a value in the interval $[a, b]$ can be calculated as

$$\Pr(a \le X \le b) = \Pr(X \in [a, b]) = \int_a^b f(x)\, dx$$
$$= \Pr(a < X \le b)$$
$$= \Pr(a \le X < b) = \Pr(a < X < b)$$



A continuous random variable assigns probability 0 to any individual value.

$$\Pr(X = a) = 0 \qquad \text{for any real number } a$$

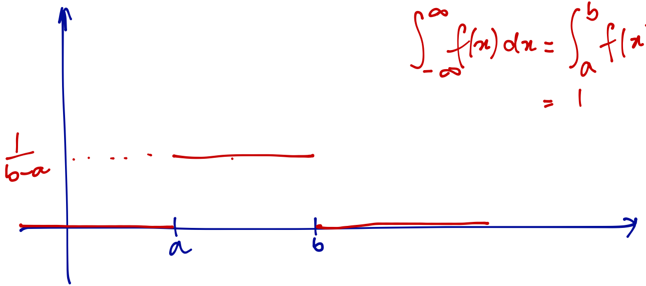
Uniform distribution on Intervals

A random variable X has a Uniform distribution on the interval $[a, b]$, if $a \le X \le b$ and the probability that X will belong to any subinterval is proportional to the length of that subinterval.

$$X \sim \text{Uniform}(a, b)$$

The pdf of a $\text{Uniform}(a, b)$ distribution is

$$f(x) = \begin{cases} \frac{1}{b-a} & \text{if } a \le x \le b \\ 0 & \text{otherwise} \end{cases}$$

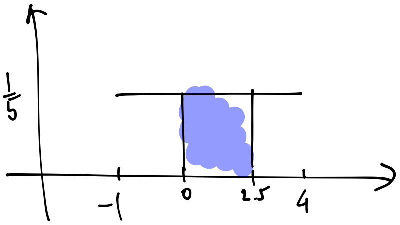


$$\int_{-\infty}^{\infty} f(x)\,dx = \int_a^b f(x)\,dx$$
$$= 1$$

Example

Say $X \sim \text{Uniform}(-1, 4)$. Write down its pdf and graph it. What is the probability that X taks a value between 0 and 2.5.

$$f(x) = \begin{cases} \frac{1}{5} & -1 \le x \le 4 \\ 0 & \text{o/w} \end{cases}$$



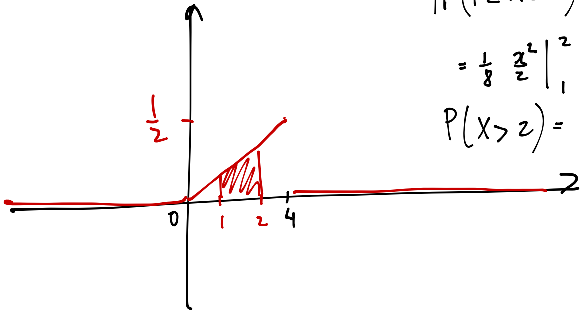
$$\Pr(0 \le X \le 2.5) = \text{shaded area}$$
$$= \int_0^{2.5} f(x)\,dx = 2.5 \cdot \frac{1}{5}$$

Example

Say X is a continuous random variable with pdf given by

$$f(x) = \begin{cases} \frac{x}{8} & \text{if } 0 < x < 4 \\ 0 & \text{otherwise} \end{cases}$$

Find $\Pr(1 \leq X \leq 2)$ and $\Pr(X > 2)$.



$$\Pr(1 \leq X \leq 2) = \int_1^2 f(x)\,dx = \int_1^2 \frac{x}{8}\,dx$$

$$= \frac{1}{8} \left. \frac{x^2}{2} \right|_1^2 = \frac{1}{8}\left(\frac{4}{2} - \frac{1}{2}\right) = \frac{3}{16}$$

$$P(X > 2) = \int_2^{\infty} f(x)\,dx = \int_2^4 f(x)\,dx$$

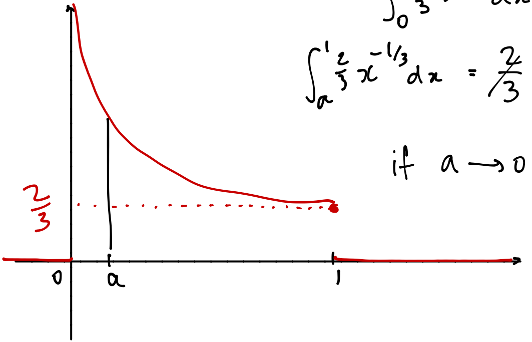
$$= \frac{1}{8} \left. \frac{x^2}{2} \right|_2^4 = \ldots$$

Example of an unbounded pdf

Say X is a continuous random variable with pdf given by

$$f(x) = \begin{cases} \frac{2}{3}x^{-1/3} & \text{if } 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

Exercise: verify that $f(x)$ is a pdf.



$$\int_0^1 \frac{2}{3} x^{-1/3} dx \stackrel{?}{=} 1$$

$$\int_a^1 \frac{2}{3} x^{-1/3} dx = \frac{\cancel{2}}{\cancel{3}} \left. \frac{x^{2/3}}{\frac{\cancel{2}}{\cancel{3}}} \right|_a^1 = 1 - a^{2/3}$$

if $a \to 0$ then $1 - a^{2/3} \to 1$

$$\int_0^1 \frac{2}{3} x^{-1/3} dx = 1$$

Example

Say X is a continuous random variable with pdf given by

$$f(x) = \begin{cases} cx^2 & \text{if } 1 \leq x < 2 \\ 0 & \text{otherwise} \end{cases}$$

1. Find the value of the constant c and sketch the pdf.
2. Find $\Pr(X > 3/2)$.

$$\int_1^2 f(x)\,dx = 1 \qquad \int_1^2 f(x)\,dx = \int_1^2 c x^2\,dx = c \cdot \frac{x^3}{3}\Big|_1^2 = c\left(\frac{8}{3} - \frac{1}{3}\right)$$

$$= c \cdot \frac{7}{3} = 1$$

$$c = \frac{3}{7}$$

$$\Pr(X > 3/2) = \int_{3/2}^2 \frac{3}{7} x^2\,dx = \frac{3}{7} \cdot \frac{x^3}{3}\Big|_{3/2}^2 = \frac{3}{7}\left(\frac{1}{3} \cdot 8 - \frac{1}{3} \cdot \frac{27}{8}\right)$$

Example

Say X is a continuous r.v. with pdf

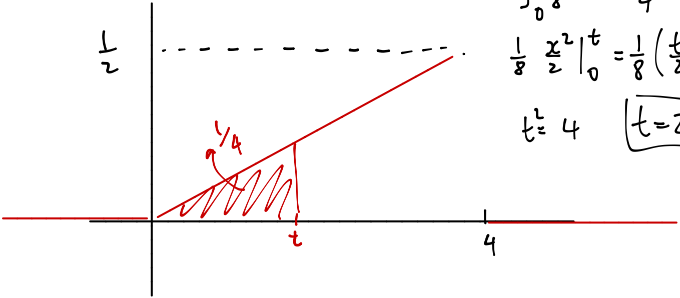
$$f(x) = \begin{cases} \frac{1}{8}x & \text{if } 0 \leq x \leq 4 \\ 0 & \text{otherwise} \end{cases}$$

1. Find the value of t such that $\Pr(X \leq t) = 1/4$
2. Find the value of t such that $\Pr(X \geq t) = 1/2$ exercise

$$\int_0^t \frac{x}{8}\,dx = \frac{1}{4}$$

$$\frac{1}{8}\left.\frac{x^2}{2}\right|_0^t = \frac{1}{8}\left(\frac{t^2}{2} - 0\right) = \frac{t^2}{16} = \frac{1}{4}$$

$$t^2 = 4 \qquad \boxed{t = 2}$$



3.3 The cumulative distribution function (cdf)

- Recall, we characterize the distribution of a r.v. X through

1. pmf .... if X is discrete.

2. pdf .... if X is continuous.

- An alternative characterization (for both cases) which is more directly related to the probabilities associated to the random variable X is obtained through the **cumulative distribution function** (**cdf**) $F(x)$.

$$F(x) = \Pr(X \leq x) \quad \text{for any real number } x$$

Bernoulli cdf

Find the cdf for a random variable $X \sim \text{Bernoulli}(p)$.

$F(x) = \Pr(X \leq x)$ for any real # x

$X \in \{0, 1\}$ $\Pr(X=1) = p$ $\Pr(X=0) = 1-p$

$F(-2.95) = \Pr(X \leq -2.95) = 0$

$F(-.8) = 0$

$\boxed{F(x) = 0 \quad \text{if } x < 0}$

$F(2.9) = \Pr(X \leq 2.9) = 1$

$F(1.0005) = 1$

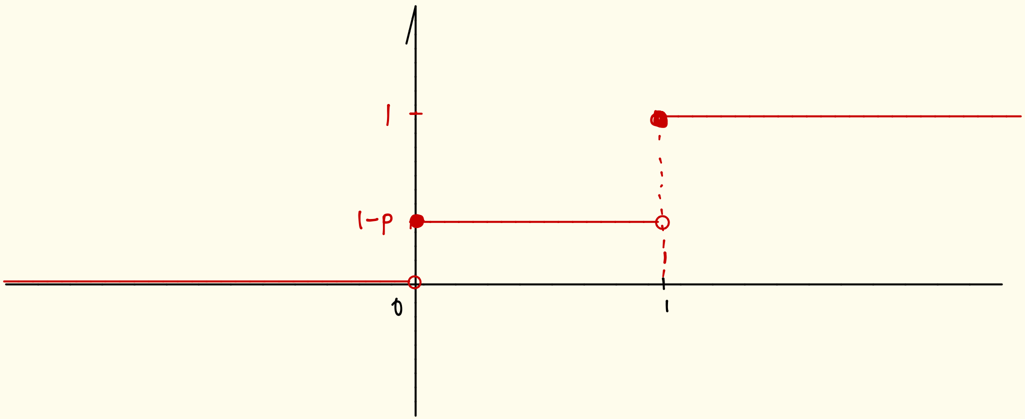
$\boxed{F(x) = 1 \quad \text{if } x > 1}$

$$F(0) = \Pr(X \le 0) = 1-p$$

$$F(0.8) = \Pr(X \le 0.8) = 1-p$$

$F(x) = 1-p$ for any $0 \le x < 1$

$$F(1) = \Pr(X \le 1) = 1$$

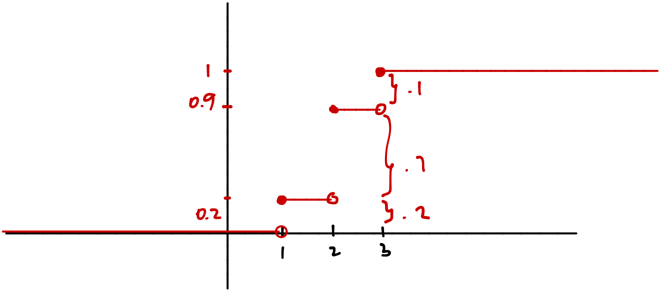


Example

Assume that X is a discrete random variable such that $X \in \{1, 2, 3\}$ with pmf given by

$$f(1) = 0.2 \quad f(2) = 0.7 \quad f(3) = 0.1$$

Derive and sketch the cdf $F(x)$.



Example

Suppose that a random variable X can take only the values 2, 0, 1, and 4, and that the probabilities of these values are as follows:

$$\Pr(X = 2) = 0.4 \quad \Pr(X = 0) = 0.1 \quad \Pr(X = 1) = 0.3 \quad \Pr(X = 4) = 0.2$$

Find and sketch the cdf of X.

Exercise

Example

Say $X \sim \mathrm{Binomial}(3, 0.1)$. Find and sketch the cdf of X.

exercise