Conditional expectations

- Let X and Y be random variables such that $E(X)$ and $E(Y)$ exist and are finite.
- The **conditional expectation** (conditional mean) of Y given that $X = x$ is defined as the expected value of the *conditional distribution* of Y given that $X = x$.

$$E(Y \mid X = x) = \int_{-\infty}^{\infty} y g_2(y \mid x) dy \quad \text{continuous case}$$

$$E(Y \mid X = x) = \sum_{\text{All } y} y g_2(y \mid x) \quad \text{discrete case}$$

Recall

$$g_2(y \mid x) = \frac{f(x, y)}{f_1(x)} \qquad \text{continuous case}$$

$$g_2(y \mid x) = \frac{\Pr(X = x, Y = y)}{\Pr(X = x)} \qquad \text{discrete case}$$

Conditional expectations

Similarly

$$E(X \mid Y = y) = \int_{-\infty}^{\infty} x g_1(x \mid y)\,dx \quad \text{continuous case}$$

$$E(X \mid Y = y) = \sum_{\text{All } x} x g_1(x \mid y) \quad \text{discrete case}$$

Recall

$$g_2(x \mid y) = \frac{f(x, y)}{f_2(y)} \quad \text{continuous case}$$

$$g_2(x \mid y) = \frac{\Pr(X = x, Y = y)}{\Pr(Y = y)} \quad \text{discrete case}$$

Example

Suppose that a point X is chosen in accordance with the uniform distribution on the interval $[0, 1]$. Also, suppose that after the value $X = x$ has been observed $(0 < x < 1)$, a point Y is chosen in accordance with a uniform distribution on the interval $[x, 1]$. Determine the value of $E(Y)$.

$$X \sim \text{Uniform}(0,1) \qquad f_1(x) = \begin{cases} 1 & x \in (0,1) \\ 0 & \text{o/w} \end{cases}$$

$$Y \mid X=x \sim \text{Uniform}(x,1) \qquad g_2(y|x) = \begin{cases} \frac{1}{1-x} & x < y < 1 \\ 0 & \text{o/w} \end{cases}$$

$$f(x,y) = f(x) \cdot g_2(y|x) = \begin{cases} \frac{1}{1-x} & 0 < x < 1 \quad x < y < 1 \\ 0 & \text{o/w} \end{cases}$$

$$E(Y) = \int_{-\infty}^{\infty} y \cdot f_2(y)\, dy$$

$$f_2(y) = \int_{-\infty}^{\infty} f(x,y)\, dx \quad = \int_0^y \frac{1}{1-x}\, dx \quad = -\log(1-x)\Big|_0^y$$

$$\log(1-x)' = -\frac{1}{1-x}$$

$$= -\log(1-y) = \log\frac{1}{1-y} \qquad y \in (0,1)$$

$$E(Y) = \int_{-\infty}^{\infty} y\, f_2(y)\, dy = \int_0^1 y \cdot \log\frac{1}{1-y}\, dy = \frac{3}{4} \quad \text{(please check!)}$$

Example

Consider a clinical trial in which a number of patients will be treated and each patient will have one of two possible outcomes: success or failure. Let P be the proportion of successes in a very large collection of patients, and let $X_i = 1$ if the ith patient is a success and $X_i = 0$ if not. Assume that the random variables $X_1, X_2, \ldots$ are conditionally independent given $P = p$ with

$$\Pr(X_i = 1 | P = p) = p$$

Let $X = X_1 + ... + X_n$, which is the number of patients out of the first n who are successes. Find the conditional mean of X given $P = p$.

$$E(X \mid P=p) = E(X_1 + X_2 + \ldots + X_n \mid P=p)$$

$$= E(X_1 \mid P=p) + E(X_2 \mid P=p) + \ldots + E(X_n \mid P=p)$$

$$= p + p + \ldots + p = np$$

Example

Suppose that X and Y have a continuous joint distribution for which the joint pdf is as follows:

$$f(x,y) = \begin{cases} x+y & \text{if } 0 \le x \le 1 \; 0 \le y \le 1, \\ 0 & \text{otherwise} \end{cases}$$

Find $E(Y \mid X = x)$ and $Var(Y \mid X = x)$.

$$E(Y \mid X = x) = \int_{-\infty}^{\infty} y \cdot g_2(y \mid x)\, dy$$

$$g_2(y \mid x) = \frac{f(x,y)}{f_1(x)} = \frac{x+y}{x+\frac{1}{2}} \qquad y \in [0,1]$$

$$f_1(x) = \int_{-\infty}^{\infty} f(x,y)\, dy = \int_0^1 (x+y)\, dy = x + \frac{1}{2}$$

$$E(Y|X=x) = \int_0^1 y \cdot \frac{x+y}{x+\frac{1}{2}}\, dy = \int_0^1 \frac{x}{x+\frac{1}{2}} \cdot y + \frac{1}{x+\frac{1}{2}} \cdot y^2 \, dy = \frac{x}{x+\frac{1}{2}} \cdot \frac{y^2}{2}\Big|_0^1 + \frac{1}{x+\frac{1}{2}} \cdot \frac{y^3}{3}\Big|_0^1$$

$$= \frac{x}{x+\frac{1}{2}} \cdot \frac{1}{2} + \frac{1}{x+\frac{1}{2}} \cdot \frac{1}{3} = A$$

$Var(Y|X=x)$ = variance of the conditional distr of Y given $X=x$

$$= \underline{E(Y^2|X=x)} - E(Y|X=x)^2 = B - A^2$$

$$E(Y^2|X=x) = \int_0^1 y^2 \cdot \frac{x+y}{x+\frac{1}{2}}\, dy = \frac{x}{x+\frac{1}{2}} \cdot \frac{y^3}{3}\Big|_0^1 + \frac{1}{x+\frac{1}{2}} \cdot \frac{y^4}{4}\Big|_0^1 = \frac{x}{x+\frac{1}{2}} \cdot \frac{1}{3} + \frac{1}{x+\frac{1}{2}} \cdot \frac{1}{4} = B$$

Functions of two or more RV

Assume that $Z = r(X, Y)$ is a random variable where the joint distribution of X, Y is described via the joint pmf/pdf $f(x, y)$. One can evaluate the conditional expectation $E(Z \mid X = x)$ in the following way

$$\begin{aligned} E(Z \mid X = x) &= E(r(X, Y) \mid X = x) \\ &= E(r(x, Y) \mid X = x) \\ &= \int_{-\infty}^{\infty} r(x, y) g_2(y \mid x) \, dy \end{aligned}$$

Example

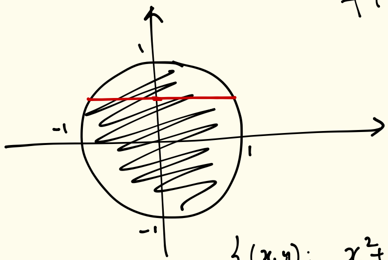
Suppose that the joint distribution of (X, Y) is uniform over the unit circle. Let $Z = X^2 + Y^2$. Find $E(Z \mid Y = -0.2)$.

$$E(Z \mid Y = y) = E(X^2 + Y^2 \mid Y = y)$$

$$= \int_{-\infty}^{\infty} (x^2 + y^2) \cdot g_2(x \mid y)\, dx$$

$$g_2(x \mid y) = \frac{f(x, y)}{f_2(y)}$$

$f(x,y) =$



$$f(x,y) = \begin{cases} \frac{1}{\pi} & x^2+y^2 \le 1 \\ 0 & o/w \end{cases}$$

$\left(\text{uniform} \approx \frac{1}{\text{area}} \right)$

$\{(x,y) : \underline{x^2+y^2 \le 1}\}$

(unit disc)

$$f_2(y) = \int_{-\infty}^{\infty} f(x,y)\,dx = \int_{-\sqrt{1-y^2}}^{+\sqrt{1-y^2}} \frac{1}{\pi}\,dx$$

$$= \frac{1}{\pi} \cdot x \Big|_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} = \frac{2}{\pi} \cdot \sqrt{1-y^2}$$

$$f_2(y) = \frac{2}{\pi}\sqrt{1-y^2} \qquad y \in [-1, 1]$$

$$g_1(x|y) = \frac{f(x,y)}{f_2(y)} = \frac{\frac{1}{\pi}}{\frac{2}{\pi} \cdot \sqrt{1-y^2}} = \frac{1}{2\sqrt{1-y^2}}$$

$$g_1(x|y) = \frac{1}{2\sqrt{1-y^2}}$$

$$E(Z|Y=y) = \int_{-\infty}^{\infty} (x^2+y^2) \cdot \frac{1}{2\sqrt{1-y^2}}\,dx = \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \frac{1}{2\sqrt{1-y^2}} (x^2+y^2)\,dx = \frac{1}{2\sqrt{1-y^2}} \left(\frac{x^3}{3} \Big|_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} + y^2 \cdot x \Big|_{-\sqrt{\cdots}}^{\sqrt{\cdots}} \right)$$

Prediction

Suppose X, Y are random variables and the goal is to predict one of them, say Y. Let d be the predicted value.

- If no information is available, the prediction that minimizes the **mean squared error (MSE)**

$$\mathrm{MSE} = E\Big((Y-d)^2\Big)$$

among all possible d values, is $d = E(Y)$.

- The prediction that minimizes the **mean absolute error (MAE)**

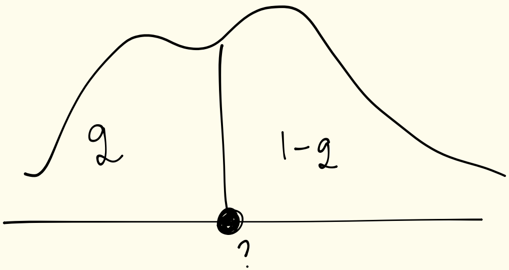
$$\mathrm{MAE} = E\Big(|\cancel{X} - d|\Big) = E\Big(|Y-d|\Big)$$

among all possible d values is $d = \mathrm{median}(\cancel{X})$.

$\mathrm{median}(Y)$

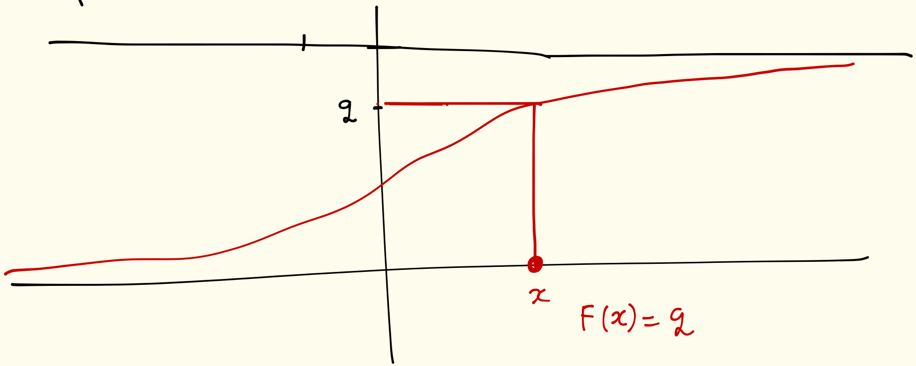
Quantiles

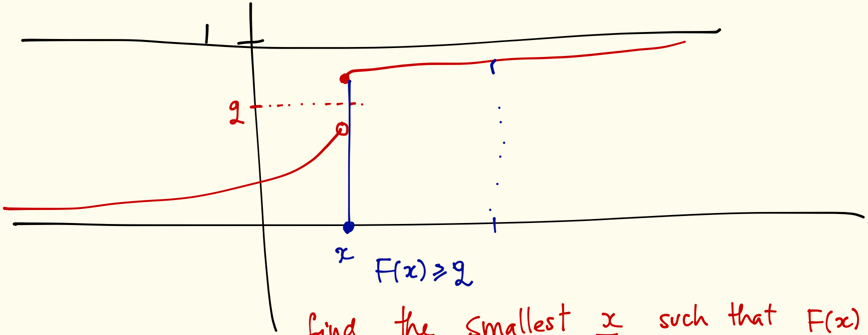
q - quantile



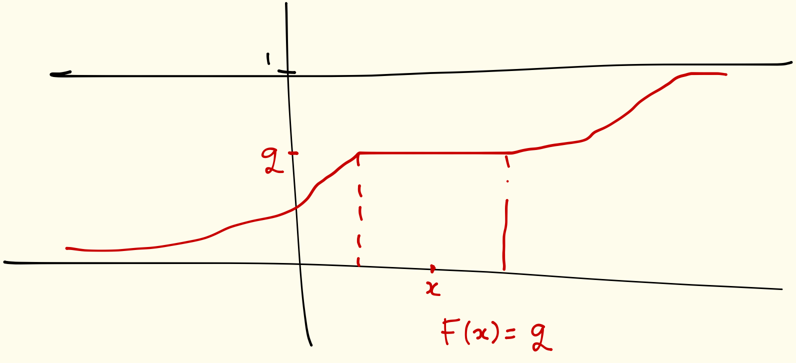
$q = \frac{1}{2}$ 50'th quantile is called the median

① find the cdf





find the smallest $\underline{\underline{x}}$ such that $F(x) \geq q$



any x is OK

choose the smallest x : $F(x) = q$