Probability density and current

The product of the wave function, $\Psi(x,t)$, and its complex conjugate, $\Psi^*(x,t)$, is the probability density for the position of a particle in *one* dimension, i.e., $\mid \Psi(x,t) \mid^2 \; dx$ yields the probability of finding a particle described by the wave function, $\Psi(x,t)$, in an infinitesimal element dx around x at a time t.[1] More explicitly, this is the probability that when measurements of the position are made on (an ensemble of) independent, identically prepared particles each of which is described by the wave function $\Psi(x,t)$, the result lies between x and $x + dx$.

In three dimensions, the wave function $\Psi(\vec{r},t)$ determines the probability density: $\mid \Psi(\vec{r},t) \mid^2 \; dxdydz$ is the probability of finding a particle in an infinitesimal volume element $dxdydz$ around $\vec{r}$.

Next we discuss the concept of a current in general; the key idea is that when something is conserved, for example, charge, then the change (increase or decrease) in the amount of a charge in a spatial region in some interval of time is equal to the amount of charge that leaves the region through the (imaginary) surface surrounding the region during that time. Charge cannot just disappear! Let $\lambda(x,t)$ be the charge density in 1 dimension ($\lambda(x,t)dx$ is the charge between $(x, x+dx)$ at time t.) The units of λ are C/m where C is Coulombs and m meters. Let us denote by $j(x,t)$ the current, the charge flowing (in the positive x direction) per unit time across x at time t. So the change in the amount of charge in the fixed interval between x and $x + dx$ in

[1]On page 2 just above boxed Equation (3) Griffiths says. " $|\Psi(x,t)|^2$ gives the probability of finding the particle at point x, at time t ...." This is corrected in the next breath but such sloppiness is reprehensible.
Remember that $|\Psi(x,t)|^2$ is the *probability density.* The probability of finding the particle at a point vanishes; only the probability of finding the particle in an interval is non-vanishing. The probability of finding the particle in an interval dx around x at time t is given by $|\Psi(x,t)|^2\, dx$ while the probability of finding a particle between a and b at a time t is given by

$$\int_a^b dx\, |\Psi(x,t)|^2 \, .$$

Also $|\Psi(x,t)|^2$ has units of 1/length in one dimension and it is dimensionally absurd to call it a probability.

the time interval $(t, t+dt)$ is given by

$$[\,\lambda(x, t+dt) \,-\, \lambda(x,t)\,]dx \quad .$$

The only way this could have occurred is for charge to have come in or out of the interval, i.e., for currents to flow. Now calculate the amount of charge which went in or out of the region (i.e., at the end points) in that time interval:

$$[\,j_x(x,t) \,-\, j_x(x+dx,t)\,]dt \quad .$$

Be absolutely sure that you understand the signs. Clearly,

$$[\,\lambda(x, t+dt) \,-\, \lambda(x,t)\,]dx \;=\; [\,j_x(x,t) \,-\, j_x(x+dx,t)\,]dt \quad .$$

The charge in the interval changes because of current flowing in or out. Expanding in a Taylor's series and retaining the lowest order terms (please do this and make sure you can) we have

$$\frac{\partial \lambda(x,t)}{\partial t} \;+\; \frac{\partial j_x(x,t)}{\partial x} \;=\; 0 \quad .$$

This is referred to as a *continuity equation* or a conservation law and appears in the context of electrodynamics (charge) and fluid dynamics (mass).

We derive the analogous result for probability in quantum mechanics and deduce other useful results below. First we find the rate of change of the probability density at x

$$\frac{\partial[\Psi(x,t)\Psi^*(x,t)]}{\partial t} \;=\; [\,\Psi^* \frac{\partial \Psi}{\partial t} \;+\; \frac{\partial \Psi^*}{\partial t}\,\Psi\,] \quad . \tag{1}$$

We will use Schrödinger's equation (and its complex conjugate) to write this in the form of a continuity equation. We have

$$i\hbar \frac{\partial \Psi}{\partial t} \;=\; -\frac{\hbar^2}{2m}\frac{\partial^2 \Psi}{\partial x^2} + V(x)\,\Psi \tag{2}$$

$$-i\hbar \frac{\partial \Psi^*}{\partial t} \;=\; -\frac{\hbar^2}{2m}\frac{\partial^2 \Psi^*}{\partial x^2} + V(x)\,\Psi^* \quad . \tag{3}$$

We multiply the first equation by Ψ^* and the second by Ψ and subtract one from the other. We obtain

$$i\hbar\frac{\partial[\Psi\Psi^*]}{\partial t} \;=\; -\frac{\hbar^2}{2m}[\,\Psi^*\frac{\partial^2\Psi}{\partial x^2} \;-\; \Psi\,\frac{\partial^2\Psi^*}{\partial x^2}\,] \quad . \tag{4}$$

Now we would like to express the right-hand side as the (negative of the) derivative with respect to x of the current. We do this as follows:

$$\frac{\partial[\Psi\Psi^*]}{\partial t} = \frac{i\hbar}{2m}\,[\,\Psi^*\,\frac{\partial^2\Psi}{\partial x^2} \;-\; \Psi\,\frac{\partial^2\Psi^*}{\partial x^2}\,] \tag{5}$$

$$= \frac{i\hbar}{2m}\,\frac{\partial}{\partial x}\,[\,\Psi^*\,\frac{\partial\Psi}{\partial x} \;-\; \Psi\,\frac{\partial\Psi^*}{\partial x}\,] \tag{6}$$

The last equality can be checked by taking the derivative canceling two terms and noting that you obtain the earlier expression. We now re-write the last line as $-\partial j/\partial x$ where we have defined the *probability current* by

$$j(x,t) \;\equiv\; \frac{\hbar}{2im}\,[\,\Psi^*\,\frac{\partial\Psi}{\partial x} \;-\; \Psi\,\frac{\partial\Psi^*}{\partial x}\,] \;\;. \tag{7}$$

The above equation enables us to derive the result in Section 1.2 that when a wave function is normalized at one instant it remains normalized and do problem 1.9. First

$$\frac{d}{dt}\int_{-\infty}^{\infty} dx\;\mid\Psi(x,t)\mid^2 \;=\; \int_{-\infty}^{\infty} dx\,\frac{\partial\mid\Psi(x,t)\mid^2}{\partial t} \tag{8}$$

$$= \int_{-\infty}^{\infty} dx\,[-\frac{\partial j}{\partial x}] \;=\; -\,[\,j(\infty,t)\;-\;j(-\infty,t)\,] \;=\; 0 \tag{9}$$

The last equality follows since we assume that there are no currents flowing in or out of the system far away. This occurs if the wave function vanishes at infinity. Thus the value of the integral is a *constant*.

We can do Problem 1.14(a); we have

$$P_{ab} \;=\; \int_a^b dx\,\mid\Psi(x,t)\mid^2 .$$

Please be clear about this: the probability of finding the particle in interval is obtained by summing (integrating) the probability to find it in each infinitesimal interval into which it can be divided. Thus

$$\frac{dP_{ab}}{dt} \;=\; \int_a^b dx\,\frac{\partial}{\partial t}\mid\Psi(x,t)\mid^2 .$$

Using the continuity equation the right-hand-side is just

$$\int_a^b dx\,\left[-\,\frac{\partial j(x,t)}{\partial x}\right]$$

and the integral of a derivative is obvious. So we obtain

$$-\,[\,j(b,t)\;-\;j(a,t)\,]\,.$$

Apart from notational differences (the book uses J instead of j) this is the required result.