The laws of classical mechanics hold for expectation values. This is one sense in which quantum mechanics can be connected to classical mechanics. This is the content of

Problem 1.7.

We show that the rate of change of the average value of the momentum is the average value of the force, the negative gradient of the potential. We start from the prescription for computing the average value of the momentum from the wave function $\Psi(x,t)$:

$$\langle p \rangle = \int dx\, \Psi^*(x,t) \left(-i\hbar \frac{\partial}{\partial x} \right) \Psi(x,t)\,.$$

We calculate its time derivative:

$$\begin{aligned} \frac{d}{dt} \langle p \rangle &= -i\hbar \frac{d}{dt} \int_{-\infty}^{\infty} dx\, \Psi^* \frac{\partial}{\partial x} \Psi \\ &= -i\hbar \int dx \, (\, \Psi^* \, \partial_x \partial_t \Psi \, + \, (\partial_t \Psi^*) \, \partial_x \Psi \,) \end{aligned}$$

We have used the product rule and interchanged the order of the partial derivatives in the preceding. For convenience I have used and will use ∂_x to denote $\partial/\partial x$ and ∂_t to denote the partial derivative with respect to time. Now use the Schrödinger equation and its complex conjugate

$$\frac{d}{dt} \langle p \rangle = -\int_{-\infty}^{\infty} dx \left[\Psi^* \, \partial_x \left(-\frac{\hbar^2}{2m} \partial_x^2 \Psi \, + \, V \Psi \right) \, + \, \left(-\frac{\hbar^2}{2m} \partial_x^2 \Psi^* \, + \, V \Psi^* \right) \partial_x \Psi \right] \quad (1)$$

Note that the derivative ∂_x in the first term acts on all the terms to its right. We simplify this[1] and obtain

$$\begin{aligned} \frac{d}{dt} \langle p \rangle &= \int_{-\infty}^{\infty} dx \, (-\Psi^* \partial_x (V \Psi) \, + \, \Psi^* V \partial_x \Psi) \\ &= \int_{-\infty}^{\infty} dx \, (-\Psi^* V \partial_x \Psi \, - \, \Psi^* (\partial_x V) \Psi \, + \, \Psi^* V \partial_x \Psi \,) \end{aligned}$$

[1]The terms that do not involve the potential cancel. Let us write them out:

$$\frac{\hbar^2}{2m} \int_{-\infty}^{\infty} \left[\Psi^* \, \partial_x^3 \Psi \, - \, \partial_x^2 \Psi^* \, \partial_x \Psi \, \right] .$$

Integrate the second term by parts twice bringing the derivatives that act on Ψ^* on Ψ. There is no change of sign (Why?). Thus they cancel.

$$
\begin{aligned}
&= \int_{-\infty}^{\infty} dx\, \Psi^* \left(-\frac{\partial V}{\partial x}\right) \Psi \text{ this is the definition of the expectation value} \\
&= \left\langle -\frac{\partial V}{\partial x} \right\rangle . \qquad (2)
\end{aligned}
$$

Thus the expectation value of the rate of change of momentum equals the expectaion value of the force given as the negative gradient of the potential.