Spin-$1/2$ dynamics

The intrinsic angular momentum of a spin-1/2 particle such as an electron, proton, or neutron assumes values $\pm\hbar/2$ along any axis. The spin state of an electron (suppressing the spatial wave function) can be described by an abstract vector or ket a concrete realization of which is a two-component column vector.

The intrinsic angular momentum of a particle is a vector operator whose components obey the standard angular momentum commutation relations. Since a spin-1/2 particle has two possible results of a measurement they can be described by 2×2 matrices. Recall the Paui representation:

$$\vec{S} \;=\; \frac{\hbar}{2}\,\vec{\sigma} \tag{1}$$

where $\vec{\sigma}$ are the Pauli spin matrices defined by

$$\sigma_x \;=\; \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_y \;=\; \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \text{and} \quad \sigma_z \;=\; \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad . \tag{2}$$

The choice makes S_z diagonal. Recall the logic of how these are determined. We can write down S_z since it is diagonal and the diagonal elements are the eigenvalues. The eigenvectors are given by

$$|z+\rangle \;=\; \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ and } |z-\rangle \;=\; \begin{pmatrix} 0 \\ 1 \end{pmatrix} .$$

We use the definitions of S_+ and S_-:

$$S_+\,|z+\rangle \;=\; 0 \;\text{ and }\; S_+\,|z-\rangle \;=\; |z+\rangle$$

$$S_-\,|z+\rangle \;=\; |z-\rangle \;\text{ and }\; S_-\,|z-\rangle \;=\; 0$$

allow us to find that

$$S_+ \;=\; \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \text{ and } S_- \;=\; \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} .$$

(Knowing S_+ we can find S_- since it is the Hermitian conjugate of S_+.) Since $S_\pm \;=\; S_x \pm iS_y$ we can find S_x and S_y.

Some simple properties that you should verify and learn to use:

$$\sigma_x^2 \;=\; \sigma_y^2 \;=\; \sigma_z^2 \;=\; I\,.$$

$$\sigma_x\sigma_y \;=\; i\sigma_z,\; \sigma_y\sigma_z \;=\; i\sigma_x,\; \sigma_z\sigma_x \;=\; i\sigma_y\,.$$

So the "standard" basis corresponds to spin up and down along the z-axis. In particular if the particle is in a state described by the ket

$$|s\rangle \;=\; \rightarrow \;\begin{pmatrix} a \\ b \end{pmatrix} \tag{3}$$

then the probability of finding $+\hbar/2$ upon making a measurement of the spin along the $z-$axis is simply $a^*a \;=\; |\,a\,|^2$. Absolutely explicitly this probability is given by the squared absolute value of the "overlap" matrix element

$$\langle z + |s\rangle \;=\; (1,\;0)\;\begin{pmatrix} a \\ b \end{pmatrix} \;=\; a \quad . \tag{4}$$

We study the effect of a magnetic field along the z-axis on a spin oriented along the x-axis initially. The solution to the more general problem follows the same logic but is algebraically more tedious. Quantum mechanically we start with the Hamiltonian for a magnetic field along the z-axis

$$H = -\gamma B_0 S_z = -\omega_L S_z = -\frac{\hbar\omega_L}{2}\sigma_z = \begin{pmatrix} -\hbar\omega_L/2 & 0 \\ 0 & \hbar\omega_L/2 \end{pmatrix} .$$

Since we are given that the spin points up along $\hat{x}$ initially we have

$$|\chi(t=0)\rangle = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} .$$

Recall that $|\chi\rangle$ is the spinor wave function that contains all the information about the system (or more precisely an ensemble of identically prepared systems). We wish to find the state at time t given by $|\chi(t)\rangle$.

Find the eigenvalues and eigenvectors of the Hamiltonian: We know that the eigenvalues of H are $\pm\hbar\omega_L/2$ since it is diagonal. The corresponding eigenvectors are

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ and } \begin{pmatrix} 0 \\ 1 \end{pmatrix} .$$

Expand the initial state in terms of the eigenfunctions of H:
We have, by inspection,[1]

$$|\chi(t-0)\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \end{pmatrix} .$$

Use the prescription for determining the state at time t by appending a factor of $e^{-iEt/\hbar}$ appropriately:
Therefore, at time t we have

$$|\chi(t)\rangle = e^{i\omega_L t/2}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + e^{-i\omega_L t/2}\frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \end{pmatrix} .$$

We find

$$|\chi(t)\rangle = \begin{pmatrix} \frac{e^{i\omega_L t/2}}{\sqrt{2}} \\ \frac{e^{-i\omega_L t/2}}{\sqrt{2}} \end{pmatrix} .$$

Given $|\chi(t)\rangle$ we can calculate expectation value of operators and the probability of making a measurement and finding a specific value. For example, suppose we wish to find the probability of measuring S_x and finding the value $+\hbar/2$. As always we find the eigenvector corresponding to the eigenvalue $+\hbar/2$ denoted by $|x+\rangle$. Then the probability of measuring $+\hbar/2$ along x at time t is given by how much the state at that time $|\chi(t)\rangle$ "looks like" the eigenvector $|x+\rangle$. This is given by the overlap $\langle x + |\chi(t)\rangle$ and the probability by $|\langle x + |\chi(t)\rangle|^2$. We can also determine expectation values in a straightforward manner as illustrated below.

[1] Please be clear about what the procedure is when it is not evident from inspection.

The expectation value of the spin along x given by $\langle\chi(t)|S_x|\chi(t)\rangle$ can be calculated. Note that if we measure the spin along the x-axis at time t on an ensemble of identically prepared (at time $t=0$) systems that evolve according to the given Hamiltonian the average value of the measurements is given by the expectation value.
Recall that $\langle\chi(t)|$ is obtained by the Hermitian conjugate operation: transpose and complex conjugate.
Therefore, we have

$$\begin{aligned}
\langle\chi(t)|S_x|\chi(t)\rangle &= \frac{\hbar}{2}\left(\frac{e^{-i\omega_L t/2}}{\sqrt{2}},\ \frac{e^{i\omega_L t/2}}{\sqrt{2}}\right)\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}\frac{e^{i\omega_L t/2}}{\sqrt{2}}\\ \frac{e^{-i\omega_L t/2}}{\sqrt{2}}\end{pmatrix}\\
&= \frac{\hbar}{2}\left(\frac{-e^{i\omega_L t/2}}{\sqrt{2}},\ \frac{e^{i\omega_L t/2}}{\sqrt{2}}\right)\begin{pmatrix}\frac{e^{-i\omega_L t/2}}{\sqrt{2}}\\ \frac{e^{i\omega_L t/2}}{\sqrt{2}}\end{pmatrix}\\
&= \frac{\hbar}{2}\frac{1}{2}\left(e^{-i\omega_L t} + e^{i\omega_L t}\right) \qquad (5)\\
&= \frac{\hbar}{2}\cos(\omega_L t)\,. \qquad (6)
\end{aligned}$$

Note that $\langle S_x\rangle(t)$ precesses in the plane perpendicular to the field. This corresponds to the classical result as we see below. Of course there is no real classical analog of spin-1/2 but we use the associated magnetic moment to investigate the effect.

Classical physics: Consider the dynamical problem of a spin-1/2 particle in a magnetic field. We will study the simplest case by choosing $\hat{z}$ along the magnetic field. We study the problem classically first. In a uniform magnetic field the magnetic moment $\vec{\mu}$ experiences a torque $\vec{\mu}\times\vec{B}$. Since the magnetic moment is proportional to the angular momentum we have $\vec{\mu} = \gamma\vec{J}$. Recall that the rate of change of the angular momentum is the torque:

$$\frac{d\vec{J}}{dt} = \vec{\mu}\times\vec{B} = \gamma\vec{J}\times\vec{B}\,. \tag{7}$$

Choosing $\vec{B} = B_0\hat{z}$ and defining $\omega_L = \gamma B_0$ we can write down the equations for each component:

$$\dot{J}_x = \omega_L J_y, \quad \dot{J}_y = -\omega_L J_x, \quad \text{and } \dot{J}_z = 0\,.$$

Clearly J_z the projection along the magnetic field is a constant in time. We solve the other two equations by a useful trick. Multiplying the equation for J_y by i and adding to the J_x equation we have

$$\dot{J}_x + i\dot{J}_y = \omega_L(J_y - iJ_x) = -i\omega_L\,(J_x + iJ_y)\,.$$

Recall that $\dot{f} = -i\omega_L f$ is easily solved as $f(t) = f(0)\,e^{-i\omega_L t}$. Check that this obeys the equation and the initial condition at $t=0$. Thus we obtain

$$J_x(t) + iJ_y(t) = (J_x(0) + iJ_y(0))\,e^{-i\omega_L t}\,.$$

Let us choose[2] $J_y(0) = 0$ so that the moment is oriented in the xz-plane initially. We have

$$J_x(t) + iJ_y(t) = J_x(0)\, e^{-i\omega_L t} \Rightarrow J_x(t) = J_x(0)\cos(\omega_L t) \text{ and } J_y(t) = -J_x(0)\sin(\omega_L t).$$

Thus we have the magnetic moment vector describing a cone with its tip moving in a circle with frequency ω_L. This is referred to as Larmor precession.

Problem: Consider a spin-1/2 particle with spin pointing up along $\hat{n}$ given by

$$\hat{n} = (\sin\theta\cos\phi,\, \sin\theta\sin\phi,\, \cos\theta)\ . \tag{8}$$

What is the 2-component vector (called a *spinor*) that corresponds to this state?

It is important to note that we are measuring the spin along $\hat{n}$ and the operator corresponding to this observable is $\vec{S}\cdot\hat{n}$. It is given by

$$\vec{S}\cdot\hat{n} = \frac{\hbar}{2}\left(\sin\theta\cos\phi\sigma_x + \sin\theta\sin\phi\sigma_y + \cos\theta\sigma_z\right) \tag{9}$$

$$\vec{S}\cdot\hat{n} = \frac{\hbar}{2}\left[\begin{pmatrix} 0 & \sin\theta\cos\phi \\ \sin\theta\cos\phi & 0 \end{pmatrix} + \begin{pmatrix} 0 & -i\sin\theta\sin\phi \\ i\sin\theta\sin\phi & 0 \end{pmatrix} + \begin{pmatrix} \cos\theta & 0 \\ 0 & -\cos\theta \end{pmatrix}\right]$$

$$\Rightarrow \quad \vec{S}\cdot\hat{n} = \frac{\hbar}{2}\begin{pmatrix} \cos\theta & \sin\theta e^{-i\phi} \\ \sin\theta e^{i\phi} & -\cos\theta \end{pmatrix}. \tag{10}$$

We need to find the eigenvalues and eigenvectors. Consider the matrix without the factor of $+\hbar/2$. Note that the trace defined to be the sum of the diagonal matrix elements. The trace is also the sum of the eigenvalues; denoting them by λ_1 and λ_2 we have $\lambda_1 + \lambda_2 = 0$. The determinant is easily calculated to be -1 and this is the product of the eigenvalues. Thus we find $\lambda_1\lambda_2 = -1$. Together we have $\lambda_1 = 1$ and $\lambda_2 = -1$. Thus the eigenvalues of $\vec{S}\cdot\hat{n}$ are $\pm\hbar/2$. This shows the result that the spin measured along any arbitrary axis yields only two possible values, $\pm\hbar/2$. This is an amazing feature of quantum mechanics. Please spend a few minutes thinking about what happens classically.

Let us denote the eigenvectors by $|\hat{n}+\rangle$ and $|\hat{n}-\rangle$. We can determine these easily:[3]

$$|\hat{n}+\rangle = \begin{pmatrix} \cos\frac{\theta}{2}\, e^{-i\phi} \\ \sin\frac{\theta}{2} \end{pmatrix} \quad \text{and} \quad |\hat{n}-\rangle = \begin{pmatrix} \sin\frac{\theta}{2}\, e^{-i\phi} \\ -\cos\frac{\theta}{2} \end{pmatrix} \tag{11}$$

[2]The general case is easily solved by choosing

$$J_x(0) + iJ_y(0) = J_\perp(0)e^{i\phi}$$

to find

$$J_x(t) = J_\perp(0)\cos(\omega_L t - \phi)) \text{ and } J_y(t) = -J_\perp(0)\sin(\omega_L t - \phi).$$

[3]We have for example

$$\frac{\hbar}{2}\begin{pmatrix} \cos\theta & \sin\theta e^{-i\phi} \\ \sin\theta e^{i\phi} & -\cos\theta \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \frac{\hbar}{2}\begin{pmatrix} a \\ b \end{pmatrix}.$$

Thus we have (canceling $\hbar/2$)

$$\cos\theta\, a + \sin\theta e^{-i\phi}\, b = a \Rightarrow \frac{a}{b} = \frac{\sin\theta}{1-\cos\theta}\, e^{-i\phi} = \frac{2\sin(\theta/2)\cos(\theta/2)}{2\sin^2(\theta/2)}e^{-i\phi} = \frac{\cos(\theta/2)}{\sin(\theta/2)}e^{-i\phi}$$

where we have used the half-angle formulae. We have chosen $a = \cos(\theta/2)e^{-i\phi}$ and $b = \sin(\theta/2)$.

This is the solution to Problem 4.31 in Griffiths (page 160) except for an overall phase factor of $\exp(i\phi)$. Feynman in Vol. III gives a more symmetrical formulae by multiplying by $e^{i\phi/2}$ (Equation 10.30):

$$|\hat{n}+\rangle \;=\; \begin{pmatrix} \cos\frac{\theta}{2}\, e^{-i\phi/2} \\ \sin\frac{\theta}{2}\, e^{i\phi/2} \end{pmatrix} \quad \text{and} \quad |\hat{n}-\rangle \;=\; \begin{pmatrix} \sin\frac{\theta}{2}\, e^{-i\phi/2} \\ -\cos\frac{\theta}{2} e^{i\phi/2} \end{pmatrix} \tag{12}$$

Problem: Given a spin in the state $|z+\rangle$, i.e., pointing up along the z-axis what are the probabilities of measuring $\pm\hbar/2$ along $\hat{n}$?

The probability of measuring up is given by $|\langle\hat{n}+|z+\rangle|^2$. This is

$$\left|(\cos(\theta/2)e^{i\phi},\, \sin(\theta/2) \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right|^2 \;=\; |\cos(\theta/2)e^{i\phi}|^2 \;=\; \cos^2(\theta/2)\,.$$

The probability of measuring $-\hbar/2$ along $\hat{n}$ given that the spin points up along z is $\sin^2(\theta/2)$. Please verify this explicitly.

How does one interpret this classically? Classically the angular momentum along $\hat{n}$ is $(\hbar/2)\cos\theta$. We have to compare the classical result with the quantum mechanical expectation value $\langle\vec{S}\cdot\hat{n}\rangle$. The expectation value or the mean values is given by the sum of the to possible values $\pm\hbar/2$ multiplied by their corresponding probabilities:

$$\frac{\hbar}{2}\cos^2(\theta/2) \,+\, \left(-\frac{\hbar}{2}\right)\sin^2(\theta/2) \;=\; \frac{\hbar}{2}\left(\cos^2(\theta/2) - \sin^2(\theta/2)\right) \;=\; \frac{\hbar}{2}\,\cos\theta\,.$$

This is an example of how expectation values conform to classical expectations in this the most quantum mechanical of systems.