

Thomson scattering, Rayleigh scattering and all that

First consider scattering of (classical) electromagnetic waves by free electrons, called *Thomson scattering*. The electron experiences a force due to the incident electric field

$$\vec{E} = \hat{\epsilon} E_0 e^{i\vec{k}\cdot\vec{r} - i\omega t}$$

where $\hat{\epsilon}$ represents the polarization direction and $\hat{\epsilon} \cdot \vec{k} = 0$. The equation of motion for the charged particle is

$$m\ddot{\vec{r}} = -e\vec{E}.$$

We recall that in the dipole approximation (valid since the wavelength of light is large compared to the scatterer) the power emitted per unit solid angle is

$$\frac{dP}{d\Omega} = \frac{e^2}{16\pi^2\epsilon_0 c^3} \langle a^2 \rangle \sin^2 \theta$$

where $\langle a^2 \rangle$ represents the time-averaged squared acceleration. The equation of motion gives the acceleration directly and the time averaged squared acceleration is

$$\langle a^2 \rangle = \frac{e^2}{2m^2} |E_0|^2.$$

This yields, upon rearranging,

$$\frac{dP}{d\Omega} = \left(\frac{e^2}{4\pi\epsilon_0 mc^2} \right)^2 \frac{\epsilon_0 c |E_0|^2}{2} \sin^2 \theta.$$

The term in the brackets is known as the *classical electron radius*:

$$\frac{e^2}{4\pi\epsilon_0 mc^2} = 2.8 \times 10^{-15} m. \quad (1.1)$$

Now recall that the time-averaged Poynting vector which is the incident energy per unit area per unit time is $I \equiv \epsilon_0 c |E_0|^2 / 2$.¹

The *differential scattering cross section* is defined by

$$\frac{d\sigma}{d\Omega} = \frac{\frac{dP}{d\Omega}}{I} \quad (1.2)$$

¹From our previous work we have

$$\vec{S} = \frac{1}{\mu_0} \vec{E} \times \vec{B} = \hat{n} \frac{c}{\mu_0} \vec{B} \cdot \vec{B} = \hat{n} \frac{1}{\mu_0 c} \vec{E} \cdot \vec{E}$$

given that $\vec{E} = c \vec{B} \times \hat{n}$. The time-averaged Poynting vector is given by

$$\langle \vec{S} \rangle = \hat{n} \frac{1}{2\mu_0 c} \vec{E} \cdot \vec{E}^* = \hat{n} \frac{c}{2\mu_0} \vec{B} \cdot \vec{B}^*.$$

In the case of plane waves the power incident per unit area per unit time is $\langle \vec{S} \rangle \cdot \hat{n} = \epsilon_0 c |E_0|^2 / 2$.

and is the area of the wavefront which delivers the same power as is scattered into a given solid angle $d\Omega$. So we obtain,

$$\frac{d\sigma}{d\Omega} = r_c^2 \sin^2 \theta . \quad (1.3)$$

The total cross section is obtained by integrating over $d\Omega$:

$$\sigma = 2\pi r_c^2 \int_0^\pi d\theta \sin \theta \sin^2 \theta = \frac{8\pi}{3} r_c^2 . \quad (1.4)$$

This is known as the classical *Thomson* cross section.

When the incident energy of the photon $\hbar\omega$ becomes comparable to the rest mass of the electron quantum mechanical effects become important. This is the case of *Compton scattering*.

Rayleigh scattering

Now consider scattering from a harmonically bound electron. In the simplest approximation we have seen before it is described by the equation of motion given below:

$$\ddot{\vec{r}} + \gamma \dot{\vec{r}} + \omega_0^2 \vec{r} = -\frac{e}{m} \vec{E}.$$

We have solved this equation before and found

$$\vec{r} = \frac{-e/m}{\omega_0^2 - i\gamma\omega - \omega^2} \vec{E}.$$

The time-averaged squared acceleration is

$$\langle \ddot{\vec{r}} \cdot \ddot{\vec{r}} \rangle = \frac{e^2}{2m^2} \left(\frac{\omega^4}{(\omega^2 - \omega_0^2)^2 + \gamma^2 \omega^2} \right).$$

Thus the scattering cross-section known as the *Rayleigh scattering cross section*² is given by

$$\sigma_R = \sigma_T \left(\frac{\omega^4}{(\omega^2 - \omega_0^2)^2 + \gamma^2 \omega^2} \right) \quad (1.5)$$

where σ_T is the Thomson scattering cross section. For strong binding (i.e., $\omega_0 \gg \omega$) we have

$$\sigma_R \approx \sigma_T \left(\frac{\omega}{\omega_0} \right)^4. \quad (1.6)$$

So the total scattering cross section is given by the inverse fourth power of the wavelength and this was derived by Rayleigh.

This formula is valid for scattering of light by air molecules (N_2 , O_2) and shows that air molecules scatter the shorter wavelengths (blue) preferentially thus leading to blueness of the sky. Explain redness of the sunset by scattering of the blue light away from the line of sight. The wavelength of red light is $650nm$ while that of blue light is $450nm$ and their ratio is 1.44. Thus blue light is scattered $1.44^4 \approx 4.3$ times as effectively as red light.

²In the first paper, Lord Rayleigh, *Phil. Mag.* **12**, 81-101 (1881) he used the concept of luminiferous ether and an elastic solid theory of the ether; he gave a derivation based on Maxwell's theory in *Phil. Mag.* **47**, 375-394 (1899).