Angular momentum in spherical coordinates

We wish to write L_x, L_y, and L_z in terms of spherical coordinates. Recall that the gradient operator is

$$\vec{\nabla} = \hat{r}\partial_r + \hat{\theta}\frac{1}{r}\partial_\theta + \hat{\phi}\frac{1}{r\sin\theta}\partial_\phi\,.$$

You should be able to write this down from a simple geometrical picture of spherical coordinates. Using $\vec{L} = -i\hbar\vec{r}\times\vec{\nabla}$ and the orthogonality of the unit vectors $\hat{r}$, $\hat{\theta}$, and $\hat{\phi}$

$$\vec{L} = (-i\hbar)\left(\hat{\phi}\,\partial_\theta - \hat{\theta}\frac{1}{\sin\theta}\partial_\phi\right)\,.$$

Since we know $\vec{L}$ in spherical coordinates we need the Cartesian unit vectors in terms of $\hat{r}$, $\hat{\theta}$, and $\hat{\phi}$:

$$\begin{aligned}
\hat{x} &= \hat{r}\,\sin\theta\,\cos\phi + \hat{\theta}\,\cos\theta\,\cos\phi - \hat{\phi}\,\sin\phi \\
\hat{y} &= \hat{r}\,\sin\theta\,\sin\phi + \hat{\theta}\,\cos\theta\,\sin\phi + \hat{\phi}\,\sin\phi \\
\hat{z} &= \hat{r}\,\cos\theta - \hat{\theta}\,\sin\theta\,.
\end{aligned}$$

Since $L_x = \hat{x}\cdot\vec{L}$ etc., we compute the scalar products and find

$$\boxed{\begin{aligned}
L_x &= -i\hbar\left(-\sin\phi\frac{\partial}{\partial\theta} - \cos\phi\cot\theta\frac{\partial}{\partial\phi}\right), \\
L_y &= -i\hbar\left(\cos\phi\frac{\partial}{\partial\theta} - \sin\phi\cot\theta\frac{\partial}{\partial\phi}\right), \\
L_z &= -i\hbar\,\frac{\partial}{\partial\phi}\,.
\end{aligned}}$$

The raising and lowering operators are given by

$$L_\pm \equiv L_x \pm iL_y = \pm\hbar\,e^{\pm i\phi}\left(\frac{\partial}{\partial\theta} \pm i\,\cot\theta\,\frac{\partial}{\partial\phi}\right)\,. \tag{1}$$

Given the cartesian components we can compute $L^2 = \vec{L}\cdot\vec{L}$. A straightforward calculation yields

$$\begin{aligned}
\vec{L}\cdot\vec{L} &= -\hbar^2\left[\frac{\partial^2}{\partial\theta^2} + \cot\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right] && (2)\\
&= -\hbar^2\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right]\,. && (3)
\end{aligned}$$

We note that the Laplacian in spherical polar coordinates is given by

$$\begin{aligned}
\nabla^2 &= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2} && (4)\\
&= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) - \frac{1}{\hbar^2 r^2}\,\vec{L}\cdot\vec{L}\,. && (5)
\end{aligned}$$

This is an important identity and you should be familiar with it (you need not remember it but you must recognize the form, be able to use it and know what its implications are.)