Wiener-Khinchin theorem

Consider a random process $x(t)$ (a random variable that evolves in time) with the autocorrelation function

$$C(\tau) \;=\; \langle x(t)\, x(t+\tau)\, \rangle\,. \tag{1}$$

x is typically thought of as voltage and the terminology stems from this identification but in general it can be any random variable of interest. The brackets denote averaging over an ensemble of realizations of the random variable, e.g., many different traces of the voltage as a function of time. We assume that the process is (weakly) stationary, i.e, the mean value $\langle x(t)\rangle$ is independent of time and the correlation function only depends on the difference of the time arguments and is independent of t in the equation above. From a practical point of view this is assumed to hold in steady state if the dynamics underlying the process is time translationally invariant. We will assume that the Fourier transform of $C(\tau)$ defined by $\boxed{\hat{C}(\omega) \;=\; \int_{-\infty}^{\infty} d\tau\, e^{-i\omega\tau}\, C(\tau)}$ exists.

Define the truncated Fourier transform of a realization of the random process $x(t)$ over an interval $[-T/2,\, T/2]$ by

$$\hat{x}_T(\omega) \;\equiv\; \int_{-T/2}^{T/2} dt\, x(t)\, e^{-i\omega t} \tag{2}$$

Since $x(t)$ is a random variable so is $\hat{x}_T(\omega)$. We define the truncated spectral power density, $S_T(\omega)$ by

$$S_T(\omega) \;\equiv\; \frac{1}{T}\,\left\langle\, |\, \hat{x}_T(\omega)\, |^2\, \right\rangle\,. \tag{3}$$

The **spectral power density** of the random process, $x(t)$ is defined by

$$S(\omega) \;=\; \lim_{T\to\infty}\, S_T(\omega) \;=\; \lim_{T\to\infty}\, \frac{1}{T}\,\left\langle\, |\, \hat{x}_T(\omega)\, |^2\, \right\rangle\,. \tag{4}$$

The **Wiener-Khinchin** theorem states (a) that the limit in Equation(4) exists and (b) the sectral power density is the Fourier transform of the autocorrelation function[1], $\boxed{S(\omega) \;=\; \int_{-\infty}^{\infty} d\tau\, C(\tau)\, e^{-i\omega\tau}}$.

Proof: Consider

$$\left\langle\, |\, \hat{x}_T(\omega)\, |^2 \,\right\rangle \;=\; \int_{-T/2}^{T/2} ds \int_{-T/2}^{T/2} dt\, \langle\, x(s)\, x(t)\, \rangle\, e^{-i\omega(s-t)} \tag{5}$$

$$\;=\; \int_{-T/2}^{T/2} ds \int_{-T/2}^{T/2} dt\, C(s-t)\, e^{-i\omega(s-t)}\,. \tag{6}$$

Since the integrand depends only on the variable $s-t$ one can do one of the standard manipulations in multivariable calculus that occurs often. Please show that for any integrable function g

$$\int_{-T/2}^{T/2} ds \int_{-T/2}^{T/2} dt\, g(s-t) \;=\; \int_{-T}^{T} d\tau\, g(\tau)\, (T \;-\; |\tau|\,)\,, \tag{7}$$

where we have defined $\tau \;\equiv\; s-t$.

Thus we obtain (identifying $g(\tau) \;=\; e^{-i\omega\tau}\, C(\tau)$)

$$\left\langle\, |\, \hat{x}_T(\omega)\, |^2 \,\right\rangle \;=\; \int_{-T}^{T} d\tau\, e^{-i\omega\tau}\, C(\tau)\;\; (\,T \;-\; |\tau|\,)\;. \tag{8}$$

At this point one can be an optimistic physicist, divide by T and let $T \;\to\; \infty$ and obtain the required result.

For a stationary process $C(\tau)$ is an even function and we have $\boxed{S(\omega) \;=\; 2\int_0^{\infty} d\tau\, C(\tau)\; \cos(\omega\tau)}$.

Note that some books and papers have a factor of 4 instead of 2. This is simply because the power spectral density for positive frequencies only, i.e., $S_>(\omega) \;=\; 2\, S(\omega)$ so that the total power remains the same. $\int_0^\infty \frac{d\omega}{2\pi}\, S_>(\omega) \;=\; \int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\, S(\omega)$.

Exercise: What is the significance of the autocorrelation function in quantum mechanics if we define it as the overlap of the wave function at times $t \;=\; 0$ and $t \;=\; \tau$, $\boxed{C(\tau) \;\equiv\; \langle\, \psi(0)|\psi(\tau)\rangle}$?

[1] This is an important result that you should remember.

Another useful representation of a random voltage function: Consider a random voltage signal $V(t)$ in a time interval $(0, T]$. This is a continuous function and we assume that the Fourier series exits and for simplicity was assume that the (time)-average value of $V(t)$ vanishes.

$$V(t) = \sum_{n=1}^{\infty} [\, a_n \cos(\omega_n t) + b_n \sin(\omega_n t)\,] \tag{9}$$

where $\boxed{\omega_n = \frac{2\pi\, n}{T}}$. Note that the $n = 0$ term has been set to zero. We consider $\{a_n\}$ and $\{b_n\}$ to be independent random variables. The coefficients vary from one voltage record to another providing an ensemble of measurements. The instantaneous power dissipated through a unit resistor is $V^2(t)$ and the time average (denoted by angular brackets) power dissipated in each Fourier component is

$$\langle \mathscr{P}_n \rangle = \frac{1}{2}\,(a_n^2 + b_n^2) \tag{10}$$

for each record. We now perform an ensemble average denoted by $\bar{\ }$ using

$$\overline{a_n} = \overline{b_n} = 0\,; \quad \overline{a_m\, b_n} = 0\,; \quad \langle a_m a_n \rangle = \langle b_m\, b_n \rangle = \sigma_n^2\, \delta_{nm}\,. \tag{11}$$

Therefore, $\boxed{\overline{\mathscr{P}_n} = \sigma_n^2}$. We also have $\boxed{\langle\, \mathscr{P}\,\rangle = \sum_n \langle\, \mathscr{P}_n \rangle = \sum_{n=1}^{\infty} \sigma_n^2}$. The random average is sufficient to make the cross terms vanish and we obtain the result without time averaging.

Consider the autocorrelation function $\boxed{C(\tau) = \overline{\langle V(t)\, V(t+\tau)\,\rangle}}$.
Exercise: use the expansion in terms of a Fourier series to establish the Wiener-Khinchin theorem.

Brownian motion

One has an individual diffusing particle constantly subjected to random collisions with the molecules of a fluid in equilibrium at a temperature T describing an erratic trajectory in space as a function of time. In 1905 the *annus mirabilis* in physics history Einstein explained it and derived the Einstein relation. Probably this paper has had a wider ranging impact from engineers to biologists and the generalizations in finance although Louis Bachelier in his thesis with Poincaré used such an equation. The key is the connection to thermodynamics and the fluctuation-dissipation theorem.

von Nägeli's argument against the molecular origin: If a molecule of mass m with speed v collides with a particle of mass M the typical momentum transfer is determined by $\boxed{M\,\Delta V \sim m\,v}$. Since $|v|$ is determined by the equipartition theorem he found $\Delta V \sim O\,(10^{-6} cm/s)$ for a particle of mass $M \sim 10^{-15}\,kg$. These changes in velocity are too small to be observed experimentally. Therefore, he concluded that Brownian motion did not arise from molecular collisions. The key is that when N molecule strike a article at random the change in velocity can be of order $\sqrt{N}$ and this yields a much larger factor. This is the essence of the random walk argument for how the sum of random variables behave.

Write a phenomenological equation of motion of a massive particle

$$M \frac{dv}{dt} = F(t) \tag{12}$$

where $F(t)$ is a random, fluctuating force due to the molecular collisions. It has a mean that can be written in the form $-\gamma M\,v$; this is the viscous drag force derived by George Gabriel Stokes for a spherical particle of radius R to be $6\pi\,\eta\,Rv$. We have $\boxed{\gamma = \dfrac{6\pi\eta\,R}{M}}$. So we write $F(t)$ as its average value $-\gamma M\,v$ and a fluctuating component $M\,\eta(t)$ to obtain the **Langevin equation**

$$\dot{v} + \gamma\,v = \eta(t)\,. \tag{13}$$

Useful definition: If an additional deterministic force F_0 is applied, then in the steady state (the derivative vanishes) then the average velocity given by $\frac{F_0}{M\gamma} \equiv \mu F_0$ where μ is defined to be the *mobility*. Thus we identify in this case $\mu = \frac{1}{M\gamma}$. Mobility is an important concept for charge carriers and electrical transport.

One can derive the Stokes-Einstein result, a fluctuation-dissipation theorem by taking averages and using the elementary identities

$$x\,\dot{x} = \frac{1}{2}\frac{d}{dt}x^2 \quad \text{and} \quad \frac{d^2}{dt^2}x^2 = 2\,\dot{x}^2 + 2\,x\,\ddot{x}\,. \tag{14}$$

Multiplying by the Langevin equation by $x(t)$ we obtain

$$x(t)\,\ddot{x} = -\gamma\,x\,\dot{x} + x(t)\,\eta(t)\,. \tag{15}$$

and using the results above we have

$$\frac{1}{2}\frac{d^2}{dt^2}x^2 - \dot{x}^2 = -\frac{\gamma}{2}\frac{d}{dt}x^2 + x(t)\,\eta(t)\,. \tag{16}$$

Take averages, use $\langle x(t)\,\eta(t)\rangle = 0$ (somewhat tricky choice) and the equipartition theorem. Defining $\frac{d\langle x^2\rangle}{dt} \equiv \alpha$ we have the differential equation

$$\dot{\alpha} + \gamma\,\alpha = \frac{2k_BT}{M}\,. \tag{17}$$

In the steady-state (at large times) we obtain

$$\langle x^2\rangle = \frac{2k_BT}{M\gamma}\,t \tag{18}$$

allowing us to identify the diffusion constant[2] $\boxed{D = \frac{k_BT}{M\gamma} = \frac{k_BT}{6\pi\eta R}}$. The Einstein relation ca be written as $\boxed{D = \mu\,k_BT}$.Note that we have used $\boxed{\mu = \frac{1}{M\gamma}}$. The accuracy of this has been confirmed to better than 1/2% in A. Westgren, *Z. Physik. Chem.* **92**, 750 (1918).

[2]Recall that $\langle\,x^2\,\rangle = 2D\,t$ in one dimension.

The random noise is assumed to be zero-mean, Gaussian, white noise for simplicity. What does this mean?[3] Zero mean implies $\boxed{\langle \eta(t) \rangle \;=\; 0}$, clearly. We then specify its autocorrelation function of the noise

$$\langle \eta(t)\, \eta(t') \,\rangle \;=\; C_1\, \delta(t - t') \tag{19}$$

where C_1 is a constant.

Exercise: What are the units/dimensions of η and C_1?

The noise is said to be delta-correlated. We will see later that $\boxed{C_1 \;=\; \dfrac{2\gamma k_B T}{M}}$. In actuality the noise is exponentially correlated with a very short time scale (molecular time scale and we are interested in the behavior of micron-sized particles) and we approximate it by a delta function for mathematical ease. Given the autocorrelation function we know that its Fourier transform yields the power spectral density - Wiener-Khinchin! Clearly, this is a constant, independent of frequency and hence is considered **white**. If the noise is not delta-correlated the noise is said to be **colored**. It is Gaussian because we say that the higher order moments are determined by the second-order moment; the first two moments determine all the moments. For example,

$$\langle \eta_1\, \eta_2\, \eta_3\, \eta_4 \rangle \;=\; \langle \eta_1 \eta_2 \rangle\, \langle \eta_3 \eta_4 \rangle \;+\; \text{other permutations}\,. \tag{20}$$

It is straightforward to see that the solution to the Langevin equation is given by

$$v(t) \;=\; v(0)\, e^{-\gamma t} \;+\; \int_0^t ds\, e^{-\gamma(t-s)}\, \eta(s)\,. \tag{21}$$

Exercise: Verify that this is a solution.

So if we compute the correlations we find (neglecting the initial conditions) we find for $\tau \;>\; 0$

$$\langle\, v(t)\, v(t+\tau) \,\rangle \;=\!\!\rightarrow\; \frac{C_1}{2\gamma}\, e^{-\gamma\tau}\,. \tag{22}$$

Exercise: Homework problem asks you to show this.

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[3] This is a stochastic ODE in the sense that one of the terms is stochastic and the solution is a random variable. The rigorous definition is the subject of the field of stochastic calculus.

For completeness we can compute

$$\langle x^2(t) \rangle \;=\; \frac{2k_BT}{M}\,\frac{\gamma\, t \;-\; (1 \;-\; e^{-\gamma t})}{\gamma^2}\,. \tag{23}$$

We also observe that

$$\int_0^\infty d\tau\,\langle\, v(0)\, v(\tau)\,\rangle \;=\; \frac{k_BT}{\gamma M} \;=\; D\,. \tag{24}$$

Thus the diffusion constant has been written as the integral of the velocity-velocity correlation function! This is an extremely important theoretical result.

The Green-Kubo formulae express transport coefficients (for example the the frequency-dependent conductivity) as integrals of (the Fourier transform, more generally, of) the appropriate current-current correlation function.

H. Nyquist, *Phys. Rev.* **32**, 110 (1928).
J. B. Johnson, *Phys. Rev*, **32**, 97 (1928).

Both Nyquist and Johnson were born in Sweden, emigrated to the United States and worked at the American Telephone and Telegraph Company, Bell Laboratories. Johnson measured it experimentally and Nyquist explained it theoretically.

In the original form Nyquist theorem states that the mean squared voltage across a resistor R in thermal equilibrium at a temperature T is given by $\boxed{\overline{V^2} \;=\; 4R\,k_B T\,\frac{\Delta\omega}{2\pi}}$, where Δf is the frequency bandwidth over which the voltage fluctuations are measured. The power spectral density is $4\,k_B T$. What does this mean? Look at voltage traces as a function of time; many samples are recorded. At a given time you average over filtered noise and compute $\langle [V^2(t)]_{filter} \rangle$ and you will find that this is $4\,k_B T\,R\,\Delta f$ where Δf is the filter bandwidth. Explain clearly why any measurement has a bandwidth. How does one derive it?

Derivation following Nyquist's engineering derivation: It is actually a one-dimensional blackbody radiation calculation. Consider a long, lossless, transmission line with a voltage source $V(t)$ and a characteristic impedance R and terminated at the other end by a load resistance R_L. This ensures no reflection and all the power is transmitted to the load.[4] Note that this means that there is no reflected wave and all the power that is transmitted down the line is absorbed by the load resistor. That is to say it acts as a black body absorbing all the radiation (electromagnetic waves) incident on it. In thermal equilibrium it maintains the same temperature by re-radiating or dissipating the incident energy. So the two can be equated.

A voltage wave of the form $V_0\,e^{ikx\,-\,i\omega t}$ propagates down the line and the condition that the voltage at $x\;=\;0$ and $x\;=\;L$ are the same yields $k\;=\;2\pi\,\frac{n}{L}$ where n is a integer. The number of modes We use a detailed balance like argument in the spirit

[4]The power dissipated in R_L is $V^2\,\frac{R_L}{(R+R_L)^2}$. Note that this is a maximum when $R\;=\;R_L$ since taking a derivative with respect to R_L yields $\frac{1}{(R+R_L)^2}\;-\;\frac{2R_L}{(R+R_L)^3}\;=\;(R+R_L)\;-\;2R_L\;=\;0$.

of Einstein ad equate the power dissipated per unit frequency by the load equals the power emitted in the range. The power incident on the load is

$$\frac{c_t}{L}\left(\frac{L}{2\pi}\frac{d\omega}{c_t}\right)\epsilon(\omega) \;=\; k_B T\,\Delta f\,. \tag{25}$$

where the second factor is the number of modes $dk/2\pi$ multiplied by the mean energy in each mode and the first term is the time it takes for the wave to travel the distance L. The right propagating modes are dissipated in the load and the left propagating modes in the impedance of the line.

Now that we have the incident power we compute the power dissipated:
The current is $\frac{V}{2R}$ and the power dissipated in a frequency range Δf around $\omega \;=\; 2\pi\, f$ is

$$R\,\langle I^2(f)\rangle\,\Delta f \;=\; \frac{1}{4R}\,\langle|\,V(\omega)\,|^2\,\rangle\,\frac{\Delta\omega}{2\pi} \;=\; \frac{1}{4R}\,S_+(\omega)\,\Delta f \tag{26}$$

where $S_+(f)$ is the power spectral density. Equating this in frequency range we have

$$S_+ \;=\; 4R\,k_B T \tag{27}$$

and the power dissipated in a frequency rage Δf is $\boxed{4R\,k_B T\,\Delta f}$.

One can do this calculation microscopically as follows. Let there be N electrons in the resistor of cross-sectional area A and length ℓ with resistivity $\rho \;=\; \frac{1}{\sigma}$. The voltage is the current (Area times current density j) divided by the resistance. The current is the charge times the total velocity, a sum of N random velocities of individual electrons:

$$V(t) \;=\; \frac{1}{R}\times A\times\frac{1}{A\ell}\,e\sum_{j=1}^{N} u_j\,. \tag{28}$$

In the presence of the voltage the electrons have an average velocity (this determines the mean current) and random fluctuations in the direction of the current due to thermal fluctuations. One can compute the voltage autocorrelation function assuming exponentially autocorrelated electron velocities and derive Nyquist's theorem.

Note about notation $S_+(\omega)$: We showed that

$$\int_{-\infty}^{\infty} d\tau \, C(\tau) \, e^{-i\omega\tau} \; = \; S(\omega) \, . \tag{29}$$

It is convenient to define the power spectral density for $\omega \geq 0$ so that $S_+(\omega) \equiv 2S(\omega)$ for positive ω and 0 for $\omega < 0$. Observe that $S(\omega) = S(-\omega)$. This is only true classically and not quantum mechanically. One way to think about this is

$$C(\tau) \; = \; \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \, S(\omega) \, e^{i\omega\tau} \; = \; \int_{0}^{\infty} \frac{d\omega}{2\pi} \, 2 \, S(\omega) \, \cos(\omega\tau) \; = \; \int_{0}^{\infty} \frac{d\omega}{2\pi} \, S_+(\omega) \, \cos(\omega\tau).$$

For a resistance of $100\,\Omega$ in a bandwidth of $1\,MHz$ we have for the root-mean-square voltage fluctuations at room temperature

$$\sqrt{k_B T \, (4\,R) \, \Delta f} \; = \; 1.28 \, \mu \, V \, . \tag{30}$$