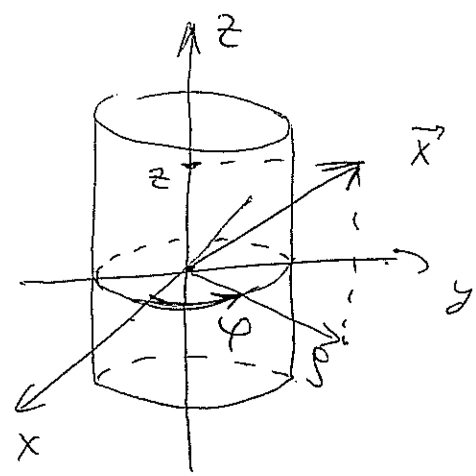
Separation of Variables and Special Functions (cont'd)

~ now we consider problems with cylindrical geometry

Separation of Variables in Cylindrical Coordinates.



Cylindrical symmetry:

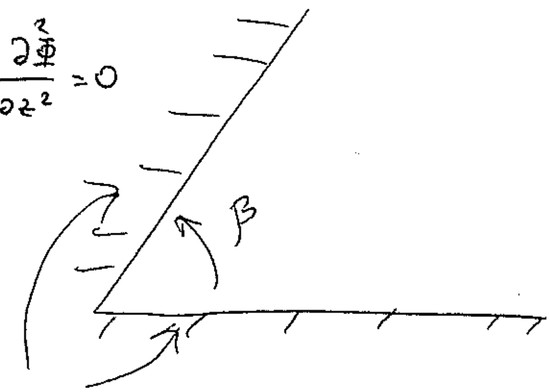
$$\begin{cases} x = \rho\cos\varphi \\ y = \rho\sin\varphi \\ z = z \end{cases}$$

A. z - independent geometry.

$$\nabla^2 \Phi = 0 \Rightarrow \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\Phi}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2\Phi}{\partial\varphi^2} + \frac{\partial^2\Phi}{\partial z^2} = 0$$

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\Phi}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2\Phi}{\partial\varphi^2} = 0$$

if $\Phi = \Phi(\rho, \varphi)$



conducting planes at potential V

Try separation of variables: $\Phi(\rho,\varphi) = R(\rho)\,\Psi(\varphi)$

$\Rightarrow$ multiplying Laplace equation by $\frac{\rho^2}{\Phi}$ we get

$$\underbrace{\frac{\rho}{R}\frac{d}{d\rho}\left(\rho\frac{dR}{d\rho}\right)}_{=\nu^2} + \underbrace{\frac{1}{\Psi}\Psi''}_{=-\nu^2} = 0$$

$$\Rightarrow \frac{\rho}{R}\frac{d}{d\rho}\left(\rho\frac{dR}{d\rho}\right) = \nu^2 \quad , \quad \frac{\Psi''}{\Psi} = -\nu^2$$

$$\Rightarrow \rho^2 R'' + \rho R' - \nu^2 R = 0$$

$$R(\rho) = \rho^\lambda \Rightarrow \lambda(\lambda-1) + \lambda - \nu^2 = 0 \Rightarrow \lambda = \pm\nu$$

$$\Rightarrow \begin{cases} R(\rho) = a\rho^\nu + b\rho^{-\nu} \\ \Psi(\varphi) = A\cos(\nu\varphi) + B\sin(\nu\varphi) \end{cases}$$

for $\nu = 0$: $\begin{cases} R(\rho) = a_0 + b_0 \ln\rho \\ \Psi(\varphi) = A_0 + B_0 \varphi \end{cases}$

General solutions of problems with cylindrical symmetry & Z-independence !

$$\Phi(\rho,\varphi) = a_0 + b_0 \ln\rho + \sum_{n=1}^{\infty}\left[a_n\rho^n \sin(n\varphi+\alpha_n) + b_n\rho^{-n}\sin(n\varphi+\beta_n)\right]$$ for $\varphi \in [0, 2\pi]$

For our problem, we want $\Phi(\varphi=0) = \Phi(\varphi=\beta) = V$

for all $\rho \Rightarrow \underbrace{b_0 = 0, \ B_0 = 0,}_{\Rightarrow a_0 A_0 = V} \quad A = 0$

no singularity at $\rho = 0 \Rightarrow b = 0$

Also, at $\varphi = \beta$: $\sin(\nu\beta) = 0 \Rightarrow \nu = \frac{\pi m}{\beta}$, $m \in$ int.

$$\Rightarrow \boxed{\Phi(\rho,\varphi) = V + \sum_{m=1}^{\infty} a_m \rho^{\frac{m\pi}{\beta}} \sin\left(\frac{m\pi}{\beta}\varphi\right)}$$

a_m's to be fixed by b.c.'s at ∞

$\Rightarrow$ need the field near $\rho \simeq 0$ $\Rightarrow m > 0 \Rightarrow$

only $m = 1$ can be kept $\Rightarrow$

$$\Phi(\rho,\varphi) \simeq V + a_1 \rho^{\frac{\pi}{\beta}} \sin\left(\frac{\pi}{\beta}\varphi\right) \quad \text{for small-}\rho$$

$$\Rightarrow E_\rho = -\frac{\partial\Phi}{\partial\rho} = -\frac{\pi a_1}{\beta}\rho^{\left(\pi/\beta - 1\right)} \sin\left(\frac{\pi}{\beta}\varphi\right)$$