Last time | Separation of Variables in Cylindrical Coordinates (cont'd)

A. z-independent geometry

$$\Phi(\rho, \varphi) = R(\rho)\,\Psi(\varphi)$$

$$\Rightarrow \begin{cases} R(\rho) = a\rho^{\nu} + b\rho^{-\nu} \\ \Psi(\varphi) = A\cos(\nu\varphi) + B\sin(\nu\varphi) \end{cases}, \quad \nu \neq 0$$

$$\begin{cases} R(\rho) = a_0 + b_0 \ln\rho \\ \Psi(\varphi) = A_0 + B_0\varphi \end{cases}, \quad \nu = 0$$

for $\varphi \in [0, 2\pi] \Rightarrow \nu =$ integer

$\Rightarrow$

$$\boxed{\Phi(\rho, \varphi) = a_0 + b_0 \ln\rho + \sum_{n=1}^{\infty} \left[a_n \rho^{n} \sin(n\varphi + \alpha_n) + b_n \rho^{-n} \sin(n\varphi + \beta_n) \right]}$$

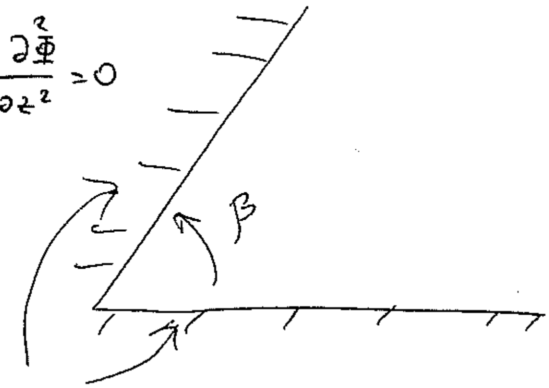
general solution of Laplace equation in cylindrical coordinates in the z-indep. geometry

A. z - independent geometry.

$$\nabla^2 \Phi = 0 \Rightarrow \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\Phi}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2\Phi}{\partial\varphi^2} + \frac{\partial^2\Phi}{\partial z^2} = 0$$

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\Phi}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2\Phi}{\partial\varphi^2} = 0$$

if $\Phi = \Phi(\rho,\varphi)$



conducting planes.
at potential V

Try separation of variables: $\Phi(\rho,\varphi) = R(\rho)\,\Psi(\varphi)$

$\Rightarrow$ multiplying Laplace equation by $\dfrac{\rho^2}{\Phi}$ we get

$$\underbrace{\frac{\rho}{R}\frac{d}{d\rho}\left(\rho\frac{dR}{d\rho}\right)}_{=\nu^2} + \underbrace{\frac{1}{\Psi}\Psi''}_{=-\nu^2} = 0$$

$$\Rightarrow \frac{\rho}{R}\frac{d}{d\rho}\left(\rho\frac{dR}{d\rho}\right) = \nu^2 \quad , \quad \frac{\Psi''}{\Psi} = -\nu^2$$

$$\Rightarrow \rho^2 R'' + \rho R' - \nu^2 R = 0$$

$$R(\rho) = \rho^\lambda \Rightarrow \lambda(\lambda-1) + \lambda - \nu^2 = 0 \Rightarrow \lambda = \pm\nu$$

$$\Rightarrow \begin{cases} R(\rho) = a\rho^\nu + b\rho^{-\nu} \\ \Psi(\varphi) = A\cos(\nu\varphi) + B\sin(\nu\varphi) \end{cases}$$

for $\nu = 0$: $\begin{cases} R(\rho) = a_0 + b_0 \ln\rho \\ \Psi(\varphi) = A_0 + B_0 \varphi \end{cases}$

General solutions of problems with cylindrical symmetry & Z-independence!

$$\Phi(\rho,\varphi) = a_0 + b_0 \ln\rho + \sum_{n=1}^{\infty}\left[a_n\rho^n \sin(n\varphi+\alpha_n) + b_n\rho^{-n}\sin(n\varphi+\beta_n)\right]$$ for $\varphi \in [0, 2\pi]$

For our problem, we want $\Phi(\varphi=0) = \Phi(\varphi=\beta) = V$

for all $\rho \Rightarrow \underbrace{b_0 = 0,\ B_0 = 0,}$ $A = 0$

$\Rightarrow a_0 A_0 = V$

no singularity at $\rho = 0 \Rightarrow b = 0$

Also, at $\varphi = \beta$: $\sin(\nu\beta) = 0 \Rightarrow \nu = \frac{\pi m}{\beta}$, $m \in$ int.

$$\Rightarrow \boxed{\Phi(\rho,\varphi) = V + \sum_{m=1}^{\infty} a_m \rho^{\frac{m\pi}{\beta}} \sin\left(\frac{m\pi}{\beta}\varphi\right).}$$

a_m's to be fixed by b.c.'s at ∞

$\Rightarrow$ need the field near $\rho \simeq 0$ $\Rightarrow m > 0 \Rightarrow$

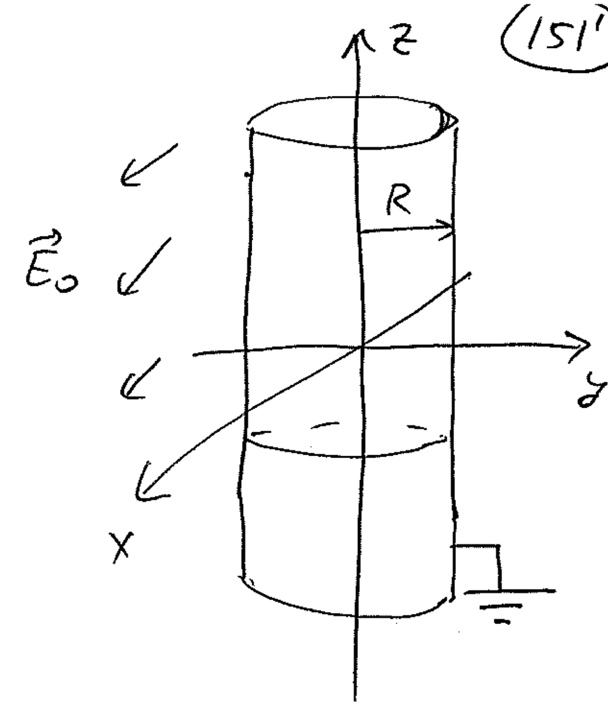
only $m=1$ can be kept $\Rightarrow$

$$\Phi(\rho,\varphi) \simeq V + a_1 \rho^{\frac{\pi}{\beta}} \sin\left(\frac{\pi}{\beta}\varphi\right) \text{ for small-}\rho$$

$$\Rightarrow E_\rho = -\frac{\partial\Phi}{\partial\rho} = -\frac{\pi a_1}{\beta}\rho^{(\pi/\beta - 1)} \sin\left(\frac{\pi}{\beta}\varphi\right)$$

Example | $\vec{E}_0 = E_0 \hat{x}$

Infinite grounded conducting cylinder of radius R in a uniform external electric field $\vec{E}_0$. Find Φ.



General solution:

$$\Phi(\rho, \varphi) = a_0 + b_0 \ln\rho + \sum_{n=1}^{\infty} \left[a_n \rho^n \sin(n\varphi + \alpha_n) + b_n \rho^{-n} \sin(n\varphi + \beta_n) \right]$$

(i) as $\rho \to \infty$ $\Rightarrow$ should get $\Phi_0 = -E_0 x = -E_0 \rho \cos\varphi$

$\Rightarrow$ require that $\Phi(\rho, \varphi)\Big|_{\rho\to\infty} = -E_0 \rho \cos\varphi$

$\Rightarrow$ $b_0 = 0$, $a_0 = 0$, $a_n = 0$ for $n \neq 1$,

$a_1 = -E_0$, $\alpha_1 = \frac{\pi}{2}$ (as $\sin(\varphi + \frac{\pi}{2}) = \cos\varphi$)

$$\Rightarrow \Phi(\rho, \varphi) = -E_0 \rho \cos\varphi + \sum_{n=1}^{\infty} b_n \rho^{-n} \sin(n\varphi + \beta_n)$$

(ii) Cylinder is grounded $\Rightarrow$ $\Phi(\rho = R, \varphi) = 0$

$$\Rightarrow -E_0 R \cos\varphi + \sum_{n=1}^{\infty} b_n R^{-n} \sin(n\varphi + \beta_n) = 0$$

$\Rightarrow \quad b_n = 0 \quad \text{for} \quad n \neq 1$

$$- E_0 R \cos\varphi + \frac{b_1}{R} \sin(\varphi + \beta_1) = 0$$

$\Rightarrow \quad \beta_1 = \frac{\pi}{2}, \quad b_1 = E_0 R^2$

$\Rightarrow$ we obtain

$$\Phi(\rho, \varphi) = - E_0 \rho \cos\varphi + \frac{E_0 R^2}{\rho} \cos\varphi$$

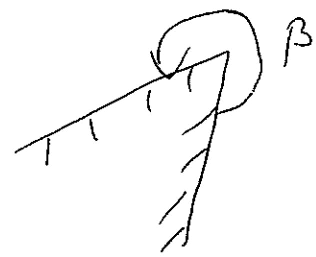
$$\Rightarrow \quad \boxed{\Phi(\rho, \varphi) = - E_0 \cos\varphi \left[\rho - \frac{R^2}{\rho} \right].}$$

$$E_\varphi = -\frac{1}{\rho}\frac{\partial \Phi}{\partial \varphi} = -\frac{\pi a_1}{\beta}\rho^{\left(\frac{\pi}{\beta}-1\right)}\cos\left(\frac{\pi}{\beta}\varphi\right).$$

$\sigma(\rho)$ at $\varphi = 0$ is $\sigma = \varepsilon_0 E_\phi(\rho, 0) \Rightarrow$

$$\boxed{\sigma = -\frac{\varepsilon_0 \pi a_1}{\beta}\rho^{\frac{\pi}{\beta}-1}} \rightarrow \infty \quad \text{as } \rho \rightarrow 0$$

if $\beta > \pi$



strong fields lead to electrical discharge $\Rightarrow$

$\Rightarrow$ leads to lightning "hitting" lightning rod!

B. Z - dependent case.

Laplace equation becomes $\frac{\partial^2}{\partial\rho^2}\Phi + \frac{1}{\rho}\frac{\partial}{\partial\rho}\Phi + \frac{1}{\rho^2}\frac{\partial^2\Phi}{\partial\varphi^2} + \frac{\partial^2\Phi}{\partial z^2} = 0$

$$\Phi(\rho, \varphi, z) = R(\rho)\, Q(\varphi)\, Z(z)$$

$$\Rightarrow R'' Q Z + \frac{1}{\rho} R' Q Z + \frac{1}{\rho^2} R Q'' Z + R Q Z'' = 0$$

divide by RQZ to obtain

$$\begin{cases} \dfrac{d^2 Z}{dz^2} - k^2 Z = 0 \\[2ex] \dfrac{d^2 Q}{d\varphi^2} + \nu^2 Q = 0 \end{cases}$$

$$\frac{d^2R}{d\rho^2} + \frac{1}{\rho}\frac{dR}{d\rho} + \left(K^2 - \frac{\nu^2}{\rho^2}\right)R = 0.$$

First two are easy: $Z = e^{\pm kz}$, $Q = e^{\pm i\nu\varphi}$

Last one is a bit tricky: first rescale: $x = k\rho$

$$\Rightarrow \frac{d^2R}{dx^2} + \frac{1}{x}\frac{dR}{dx} + \left(1 - \frac{\nu^2}{x^2}\right)R = 0$$

Look for solution in the form of a series:

$$R(x) = x^{\alpha}\sum_{j=0}^{\infty} a_j x^j \quad , \quad a_0 \neq 0$$

$$\Rightarrow \sum_{j=0}^{\infty}\Big[a_j \cdot (j+\alpha)(j+\alpha-\cancel{1})\, x^{j-2+\alpha} + a_j\cdot(j+\alpha)\cdot x^{\alpha+j-2} - \nu^2 a_j x^{\alpha+j-2}\Big] + \sum_{j=0}^{\infty} a_j x^{j+\alpha} = 0$$

$j=0$: $\quad a_0\,(\alpha^2-\nu^2)\cdot x^{\alpha-2} = 0 \quad \Rightarrow$ if $a_0 \neq 0 \Rightarrow \alpha = \pm\nu$.

1st non-vanishing coef.

$j=1$: $\quad a_1\left[(\alpha+1)^2 - \nu^2\right] x^{\alpha-1} = 0 \quad \Rightarrow a_1 = 0.$

other j's: $\quad a_{j+2}\left[(j+\alpha+2)^2 - \nu^2\right] + a_j = 0 \quad \Big|$ coef. at $x^{j+\alpha}$

$$\Rightarrow a_{j+2} = -a_j \frac{1}{(j+2)^2 + 2\alpha(j+2)} \Rightarrow$$