Last time | We worked out two examples on solving Laplace equation for cylindrical geometry in the z-independent case.

B. z-dependent geometry.

$$\Phi(\rho, \varphi, z) = R(\rho)\, Q(\varphi)\, Z(z)$$

& plugged into $\nabla^2 \Phi = 0$.

We got

$$\begin{cases} Z'' - k^2 Z = 0 & \Rightarrow\ Z(z) = e^{\pm kz} \\ Q'' + \nu^2 Q = 0 & \Rightarrow\ Q(\varphi) = e^{\pm i\nu\varphi} \\ R'' + \frac{1}{\rho} R' + \left(k^2 - \frac{\nu^2}{\rho^2}\right) R = 0 & \end{cases}$$

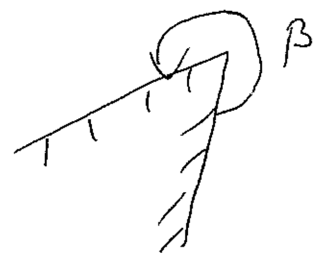
$\Rightarrow$ now we need to solve the equation for R.

$$E_\varphi = -\frac{1}{\rho}\frac{\partial \Phi}{\partial \varphi} = -\frac{\pi a_1}{\beta}\rho^{\left(\frac{\pi}{\beta}-1\right)}\cos\left(\frac{\pi}{\beta}\varphi\right).$$

$\sigma(\rho)$ at $\varphi = 0$ is $\sigma = \varepsilon_0 E_\phi(\rho, 0) \Rightarrow$

$$\boxed{\sigma = -\frac{\varepsilon_0 \pi a_1}{\beta}\rho^{\frac{\pi}{\beta}-1}} \rightarrow \infty \quad \text{as } \rho \rightarrow 0$$

if $\beta > \pi$



strong fields lead to electrical discharge $\Rightarrow$

$\Rightarrow$ leads to lightning "hitting" lightning rod!

B. z-dependent case.

Laplace equation becomes $\frac{\partial^2}{\partial \rho^2}\Phi + \frac{1}{\rho}\frac{\partial}{\partial \rho}\Phi + \frac{1}{\rho^2}\frac{\partial^2 \Phi}{\partial \varphi^2} + \frac{\partial^2 \Phi}{\partial z^2} = 0$

$$\Phi(\rho, \varphi, z) = R(\rho)\,Q(\varphi)\,Z(z)$$

$$\Rightarrow \quad R''QZ + \frac{1}{\rho}R'QZ + \frac{1}{\rho^2}RQ''Z + RQZ'' = 0$$

divide by RQZ to obtain

$$\begin{cases} \dfrac{d^2 Z}{dz^2} - k^2 Z = 0 \\[2ex] \dfrac{d^2 Q}{d\varphi^2} + \nu^2 Q = 0 \end{cases}$$

$$\frac{d^2R}{d\rho^2} + \frac{1}{\rho}\frac{dR}{d\rho} + \left(K^2 - \frac{\nu^2}{\rho^2}\right) R = 0.$$

First two are easy: $Z = e^{\pm kz}$, $Q = e^{\pm i\nu\varphi}$

Last one is a bit tricky: first rescale: $x = k\rho$

$$\Rightarrow \frac{d^2R}{dx^2} + \frac{1}{x}\frac{dR}{dx} + \left(1 - \frac{\nu^2}{x^2}\right)R = 0$$

Look for solution in the form of a series:

$$R(x) = x^\alpha \sum_{j=0}^{\infty} a_j x^j, \quad a_0 \neq 0$$

$$\Rightarrow \sum_{j=0}^{\infty} \Big[a_j \cdot (j+\alpha)(j+\alpha-\cancel{1}) x^{j-2+\alpha} + a_j \cdot (j+\alpha) \cdot x^{\alpha+j-2} - \nu^2 a_j x^{\alpha+j-2} \Big] + \sum_{j=0} a_j x^{j+\alpha} = 0$$

$j=0$: $\quad a_0 (\alpha^2 - \nu^2) \cdot x^{\alpha-2} = 0 \Rightarrow$ if $a_0 \neq 0 \Rightarrow \alpha = \pm\nu$. (1st non-vanishing coef.)

$j=1$: $\quad a_1 [(\alpha+1)^2 - \nu^2] x^{\alpha-1} = 0 \Rightarrow a_1 = 0.$

other j's: $\quad a_{j+2}\left[(j+\alpha+2)^2 - \nu^2\right] + a_j = 0 \quad \Big|$ coef. at $x^{j+\alpha}$

$$\Rightarrow a_{j+2} = -a_j \frac{1}{(j+2)^2 + 2\alpha(j+2)} \Rightarrow$$

$$\Rightarrow \boxed{a_j = -\frac{a_{j-2}}{j \cdot (j+2\alpha)}}$$

$$\Rightarrow a_{2n+1} = 0 \quad \text{for } \forall \text{ integer } n$$

$$a_{2j} = -\frac{a_{2j-2}}{4j(j+\alpha)}$$

Convention: choose $a_0 = \dfrac{1}{2^\alpha \Gamma(\alpha+1)}$

$$\Rightarrow a_{2j} = (-1)^j \frac{1}{4^j} \frac{\cancel{\alpha!}}{j!\,(j+\alpha)!} \frac{1}{2^\alpha \cancel{\Gamma(\alpha+1)}}$$

$$\Rightarrow a_{2j} = (-1)^j \frac{1}{2^{2j+\alpha}\,\Gamma(j+1)\,\Gamma(j+\alpha+1)}$$

$\Rightarrow$ for $\boxed{\alpha = \pm\nu}$ we get 2 different solutions:

$$\boxed{\begin{aligned} J_\nu(x) &= \left(\frac{x}{2}\right)^\nu \sum_{j=0}^{\infty} \frac{(-1)^j}{j!\,\Gamma(j+\nu+1)} \left(\frac{x}{2}\right)^{2j} \\ J_{-\nu}(x) &= \left(\frac{x}{2}\right)^{-\nu} \sum_{j=0}^{\infty} \frac{(-1)^j}{j!\,\Gamma(j-\nu+1)} \left(\frac{x}{2}\right)^{2j} \end{aligned}}$$

Bessel functions of the 1st kind.

example: rotating rope  $\sim J_0\left(2\omega\sqrt{\frac{z}{g}}\right)$.

Two solutions are independent if ν is not an integer! For interger $\nu = m$:

$$J_{-m}(z) = (-1)^m J_m(z) \quad \text{related.}$$

To avoid it define Neumann function (Bessel function of the 2nd kind)

$$N_\nu(x) = \frac{J_\nu(x)\cos(\nu\pi) - J_{-\nu}(x)}{\sin(\nu\pi)}$$

$\nu \to$ integer, $N_\nu(x)$ behaves fine.

Gamma function: $\Gamma(z) \equiv \int_0^\infty t^{z-1} e^{-t}\, dt$

well defined for $z > 0$, analytically continue to $z \leq 0$ using $z\Gamma(z) = \Gamma(1+z)$.

$\Gamma(1+m) = m!$ for integer m.

Recursion relations (can be easily proven) (series repr.)

$$\begin{cases} J_{\nu-1}(z) + J_{\nu+1}(z) = \dfrac{2\nu}{z} J_\nu(z) \\ J_{\nu-1}(z) - J_{\nu+1}(z) = 2\dfrac{dJ_\nu(z)}{dz} \end{cases}$$

work for $N_\nu(z)$ as well

e.g. for $\nu = 0$ $\quad J_{-1}(z) - J_1(z) = 2\, J_0'(z)$

as $J_{-1}(z) = -J_1(z) \Rightarrow \quad J_1(z) = -J_0'(z)$.

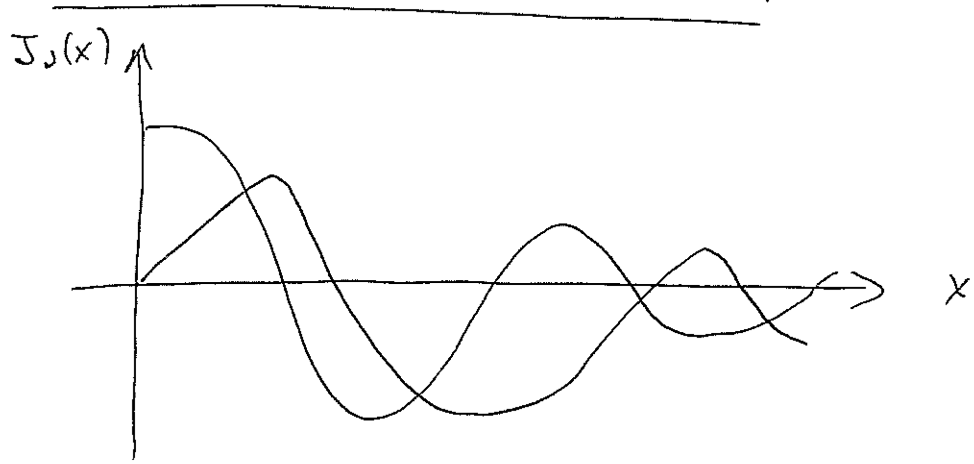
Asymptotics: small x:

$$\begin{cases} J_\nu(x) \to \dfrac{1}{\Gamma(\nu+1)} \left(\dfrac{x}{2}\right)^\nu & \text{if } x \ll 1 \\ & \text{(1st term in the series)} \\ N_\nu(x) \to \begin{cases} \dfrac{2}{\pi}\left[\ln \dfrac{x}{2} + 0.5772\ldots\right], & \nu = 0 \\ -\dfrac{\Gamma(\nu)}{\pi}\left(\dfrac{2}{x}\right)^\nu, & \nu \neq 0 \end{cases} & \text{if } x \to 0. \end{cases}$$

large x:

$$\begin{cases} J_\nu(x) \to \sqrt{\dfrac{2}{\pi x}} \cos\left(x - \dfrac{\nu\pi}{2} - \dfrac{\pi}{4}\right), & x \gg 1 \\ N_\nu(x) \to \sqrt{\dfrac{2}{\pi x}} \sin\left(x - \dfrac{\nu\pi}{2} - \dfrac{\pi}{4}\right), & x \gg 1. \end{cases}$$

Bessel function roots:



$J_\nu \sim x^\nu$ when $x \to 0$

$$\Rightarrow \left.\begin{aligned} J_0(0) &= 1 \\ J_1(x) &\sim x \\ J_2(x) &\sim x^2 \end{aligned}\right\} \text{ for } x \ll 1.$$