Last time roots of Bessel functions:

$$J_\nu(x_{\nu n}) = 0 \ , \quad n = 1, 2, 3, \ldots$$

$x_{\nu n}$ is the nth root of Bessel function $J_\nu(x)$

⇝ Bessel functions form a complete & orthonormal set

$$\int_0^a d\rho \cdot \rho \cdot J_\nu\left(x_{\nu n} \frac{\rho}{a}\right) J_\nu\left(x_{\nu m} \frac{\rho}{a}\right) = \frac{a^2}{2} \delta_{nm} \left[J_{\nu+1}(x_{\nu n})\right]^2$$

on the set of functions $f(\rho)$, $\rho \in [0, a]$, $f(0) = 0 = f(a)$.

Defined modified Bessel functions:

$$\begin{cases} I_\nu(x) \equiv i^{-\nu} J_\nu(ix) \\ K_\nu(x) \equiv \frac{\pi}{2} i^{\nu+1} \left[J_\nu(ix) + i N_\nu(ix)\right] \end{cases}$$

Note that: $$I_\nu(x)\Big|_{x \ll 1} \simeq \frac{1}{\Gamma(\nu+1)} \left(\frac{x}{2}\right)^\nu$$

$$K_\nu(x)\Big|_{x \ll 1} \simeq \begin{cases} -\ln\frac{x}{2} - 0.5772\ldots \ , & \nu = 0 \\ \frac{\Gamma(\nu)}{2}\left(\frac{2}{x}\right)^\nu \ , & \nu \neq 0 \end{cases}$$

$$I_\nu(x)\Big|_{x \gg 1} \simeq \frac{1}{\sqrt{2\pi x}} e^x$$

$$K_\nu(x)\Big|_{x \gg 1} \simeq \sqrt{\frac{\pi}{2x}} e^{-x}$$

Another useful special functions are
modified Bessel functions $I_\nu(z)$ & $K_\nu(z)$

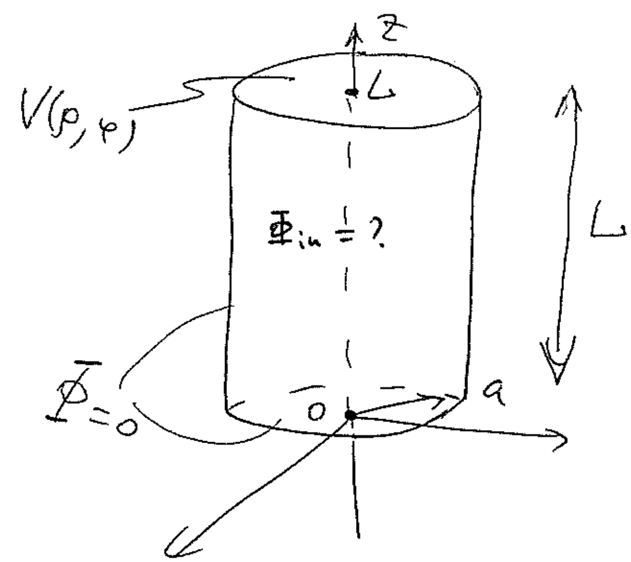
Definition $\qquad I_\nu(x) = i^{-\nu} J_\nu(ix)$

$$K_\nu(x) = \frac{\pi}{2} i^{\nu+1} \left[J_\nu(ix) + i N_\nu(ix) \right]$$

obey $\quad \frac{d^2R}{dx^2} + \frac{1}{x}\frac{dR}{dx} - \left(1 + \frac{\nu^2}{x^2}\right) R = 0$ diff. equation.

Very useful formula: $\boxed{\frac{1}{k}\,\delta(k-k') = \int_0^\infty dx \cdot x \cdot J_\nu(kx)\, J_\nu(k'x)}$

Example of a Boundary - Value Problem:



$Q(\varphi) = A\sin(m\varphi) + B\cos(m\varphi)$

$Z(z) = C\sinh(kz) + D\cosh(kz)$

$\Rightarrow D = 0$ as $Z(0) = 0$.

$R(\rho) = E\, J_m(k\rho) + F N_m(k\rho)$

$\Rightarrow$ finite at $\rho = 0 \Rightarrow F = 0$

$R(a) = 0 \Rightarrow \quad k \to k_{mn} = \frac{x_{mn}}{a}$ ~ roots

$$\Rightarrow \Phi(\rho, \varphi, z) = \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} J_m(k_{mn}\rho)\sinh(k_{mn} z) \cdot \left[A_{mn}\sin(m\varphi) + B_{mn}\cos(m\varphi) \right].$$

Finally, $\Phi(\rho, \varphi, z = L) = V(\rho, \varphi)$

$$\Rightarrow V(\rho,\varphi) = \sum_{m,n} J_m(k_{mn}\rho)\sinh(k_{mn}L)\left[A_{mn}\sin(m\varphi) + B_{mn}\cos(m\varphi)\right] \Rightarrow$$

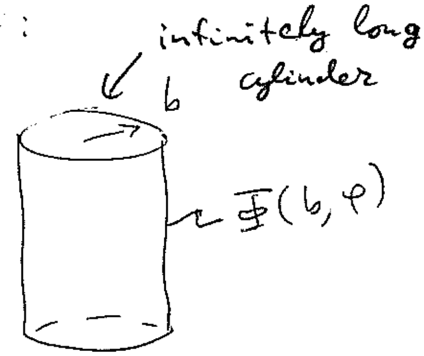
invert the Fourier (φ) and Fourier - Bessel (ρ) series to get

$$\begin{pmatrix} A_{mn} \\ B_{mn} \end{pmatrix} = \frac{2}{\pi a^2 \sinh(k_{mn}L)\, J_{m+1}^2(k_{mn}a)} \int_0^{2\pi} d\varphi \cdot \int_0^a d\rho \cdot \rho \cdot V(\rho,\varphi)\, J_m(k_{mn}\rho) \begin{pmatrix} \sin(m\varphi) \\ \cos(m\varphi) \end{pmatrix}.$$

for $m=0$ use $2B_{0n}$ on the right-hand side
(that is, divide B_{0n} from the f-la by 2 to obtain true B_{0n})

<u>Jackson problem 2.12</u>:

$$\Phi(\rho,\varphi) = a_0 + b_0 \ln\rho + \sum_{n=1}^{\infty} a_n \rho^n \sin(n\varphi + \alpha_n) + \sum_{n=1}^{\infty} b_n \rho^{-n} \sin(n\varphi + \beta_n)$$



$\Rightarrow b_n = 0$ as Φ is finite at $\rho = 0$ ($b_0 = 0$ too)

$$\Rightarrow \Phi(\rho,\varphi) = \frac{A_0}{2} + \sum_{m=1}^{\infty} \left(A_m\cos(m\varphi) + B_m \sin(m\varphi)\right)\rho^m$$

For $\rho = b$ we have

$$\Phi(b,\varphi) = \frac{A_0}{2} + \sum_{m=1}^{\infty} \left(A_m \cos(m\phi) + B_m \sin(m\varphi) \right) b^m$$

$$\Rightarrow \begin{pmatrix} A_m \\ B_m \end{pmatrix} b^m = \frac{1}{\pi} \int_0^{2\pi} d\varphi' \begin{pmatrix} \cos(m\varphi') \\ \sin(m\varphi') \end{pmatrix} \Phi(b,\varphi'), \quad A_0 = \frac{1}{\pi}\int_0^{2\pi} d\varphi' \Phi(b,\varphi')$$

$$\Rightarrow \Phi(\rho,\varphi) = \frac{1}{\pi}\int_0^{2\pi} d\varphi' \, \Phi(b,\varphi') \sum_{m=1}^{\infty} \left[\cos m\varphi \cos m\varphi' + \right.$$

$$\left. + \sin m\varphi \sin m\varphi' \right] \cdot b^{-m} \rho^m + \frac{1}{2\pi}\int_0^{2\pi} d\varphi' \, \Phi(b,\varphi').$$

Now, $[\quad] = \cos m(\varphi - \varphi') = \frac{1}{2}\left[e^{im(\varphi-\varphi')} + e^{-im(\varphi-\varphi')} \right]$

$$\Rightarrow \sum_{m=1}^{\infty} e^{im(\varphi-\varphi')} \left(\frac{\rho}{b}\right)^m = \frac{\rho}{b} e^{i(\varphi-\varphi')} \frac{1}{1 - \frac{\rho}{b} e^{i(\varphi-\varphi')}} =$$

$$= \frac{1}{\frac{b}{\rho} e^{-i(\varphi-\varphi')} - 1} \Rightarrow \sum_{m=1}^{\infty} \left(\frac{\rho}{b}\right)^m \cos m(\varphi-\varphi') =$$

$$= \frac{1}{2}\left[\frac{1}{\frac{b}{\rho} e^{-i(\varphi-\varphi')} - 1} + \frac{1}{\frac{b}{\rho} e^{i(\varphi-\varphi')} - 1} \right] = \frac{\frac{b}{\rho}\cos(\varphi-\varphi') - 1}{1 + \frac{b^2}{\rho^2} - 2\frac{b}{\rho}\cos(\varphi-\varphi')}$$

$$= \frac{b\rho\cos(\varphi-\varphi') - \rho^2}{\rho^2 + b^2 - 2\rho b \cos(\varphi-\varphi')}$$

So, $\Phi(\rho,\varphi) = \frac{1}{2\pi}\int_0^{2\pi} d\varphi' \, \Phi(b,\varphi') \left[1 + 2\frac{b\rho\cos(\varphi-\varphi') - \rho^2}{b^2 + \rho^2 - 2\rho b\cos(\varphi-\varphi')} \right]$

$$= \frac{1}{2\pi} \int_0^{2\pi} d\varphi' \, \Phi(b, \varphi') \frac{b^2 - \rho^2}{b^2 + \rho^2 - 2b\rho\cos(\varphi - \varphi')}$$ as desired!

Green function in cylindrical coordinates:

need to solve $\nabla^2 G(\vec{x}, \vec{x}\,') = -4\pi\delta^3(\vec{x} - \vec{x}\,') =$

$$= -\frac{4\pi}{\rho}\delta(\rho - \rho')\,\delta(\varphi - \varphi')\,\delta(z - z').$$

write $\delta(z - z') = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk \, e^{ik(z - z')} = \int_0^{\infty} \frac{dk}{\pi}\cos[k(z - z')]$

$$\delta(\varphi - \varphi') = \frac{1}{2\pi}\sum_{m=-\infty}^{\infty} e^{im(\varphi - \varphi')}$$

$$\Rightarrow G(\vec{x}, \vec{x}\,') = \frac{1}{2\pi^2}\sum_{m=-\infty}^{\infty}\int_0^{\infty} dk \, e^{im(\varphi - \varphi')}\cos[k(z - z')] \cdot g_m(k, \rho, \rho')$$

Plug this back into eqn for G to get

$$\frac{1}{\rho}\frac{d}{d\rho}\left(\rho\frac{d}{d\rho}g_m(k, \rho, \rho')\right) - \left(k^2 + \frac{m^2}{\rho^2}\right)g_m = -\frac{4\pi}{\rho}\delta(\rho - \rho')$$

Same story as in rectangular coord's:

if $\rho < \rho'$ $\Rightarrow$ get $g_m \sim A I_m(k\rho) + B K_m(k\rho)$

$\Rightarrow$ we want regular behavior as $\rho \to 0$ $\Rightarrow$

$\Rightarrow g_m \sim I_m(k\rho)$ for $\rho < \rho'$