Last time | Constructed the Green function in cylindrical coordinates: of Laplace operator

$$G(\vec{x},\vec{x}') = \frac{1}{|\vec{x}-\vec{x}'|} = \frac{2}{\pi}\sum_{m=-\infty}^{\infty}\int_0^\infty dk\, e^{im(\varphi-\varphi')}\cos[k(z-z')] \cdot I_m(k\rho_<)\, K_m(k\rho_>) .$$

Defined the Wronskian

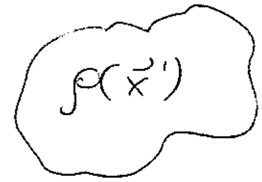
$$W[u,v] \equiv uv' - vu' = \begin{vmatrix} u & v \\ u' & v' \end{vmatrix}.$$

and argued that

$$W[I_\nu(x), K_\nu(x)] = -\frac{1}{x}$$

Multipole Expansion.

$$\Phi(\vec{x}) = \frac{1}{4\pi\varepsilon_0}\int d^3x' \, \frac{\rho(\vec{x}')}{|\vec{x}-\vec{x}'|}$$



localized charges

Use

$$\frac{1}{|\vec{x}-\vec{x}'|} = 4\pi \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} \frac{1}{2\ell+1} \frac{r_<^{\ell}}{r_>^{\ell+1}} Y_{\ell m}^*(\theta',\varphi') Y_{\ell m}(\theta,\varphi).$$

to obtain

$$\Phi(\vec{x}) = \frac{1}{\varepsilon_0}\sum_{\ell,m}\frac{1}{2\ell+1}\left[\int d^3x' \, Y_{\ell m}^*(\theta',\varphi') r'^{\ell} \rho(\vec{x}')\right] \cdot$$

$$\cdot Y_{\ell m}(\theta,\varphi) \cdot \frac{1}{r^{\ell+1}}$$, where we've used the fact that far from the charges $r_> = r$, $r_< = r'$

Definition Defining <u>multipole moments</u>

$$\boxed{q_{\ell m} = \int d^3x' \, Y_{\ell m}^*(\theta',\varphi') r'^{\ell} \rho(\vec{x}')}$$

we obtain

$$\boxed{\Phi(\vec{x}) = \frac{1}{\varepsilon_0}\sum_{\ell,m}\frac{1}{2\ell+1}\frac{q_{\ell m}}{r^{\ell+1}} Y_{\ell m}(\theta,\varphi)}$$

multipole expansion.

usefull at large r, where the expansion parameter $\sim 1/r$ is small $\Rightarrow$ can approximate Φ by the first few terms in the series

Some low-order multipole moments:

$$Y_{00} = \frac{1}{\sqrt{4\pi}} \Rightarrow \quad q_{00} = \frac{1}{\sqrt{4\pi}} \int d^3x' \rho(\vec{x}') = \boxed{\frac{q}{\sqrt{4\pi}} = q_{00}}$$

where q is the total charge in the system.

$$Y_{11} = -\sqrt{\frac{3}{8\pi}} \sin\theta \overbrace{(\cos\varphi + i\sin\varphi)}^{e^{i\varphi}} = -\sqrt{\frac{3}{8\pi}} \frac{x+iy}{r}$$

$$Y_{10} = +\sqrt{\frac{3}{4\pi}} \cos\theta = \sqrt{\frac{3}{4\pi}} \frac{z}{r}$$

as $Y_{\ell m}^{*} \leftarrow$ c.c.

$$\Rightarrow q_{11} = -\sqrt{\frac{3}{8\pi}} \int d^3x' \rho(\vec{x}') \cdot (x' - iy') =$$

$$= -\sqrt{\frac{3}{8\pi}} (p_x - ip_y)$$

$$q_{10} = \sqrt{\frac{3}{4\pi}} \int d^3x' \rho(\vec{x}') z' = \sqrt{\frac{3}{4\pi}} p_z$$

where we defined electric dipole moment (EDM)

$$\boxed{\vec{p} \equiv \int d^3x' \rho(\vec{x}') \vec{x}'}$$

$$Y_{22} = \frac{1}{4}\sqrt{\frac{15}{2\pi}} \sin^2\theta e^{2i\varphi} = \frac{1}{4}\sqrt{\frac{15}{2\pi}} \sin^2\theta (\cos^2\varphi -$$

$$- \sin^2\varphi + 2i \sin\varphi \cos\varphi) = \frac{1}{4r^2}\sqrt{\frac{15}{2\pi}} (x^2 - y^2 + 2ixy)$$

$$\Rightarrow q_{22} = \frac{1}{4}\sqrt{\frac{15}{2\pi}} \int d^3x' \rho(\vec{x}') (x' - iy')^2 =$$

$$= \frac{1}{12}\sqrt{\frac{15}{2\pi}}\left(Q_{11} - 2i\,Q_{12} - Q_{22}\right)$$

where we've defined (traceless) quadrupole moment tensor

$$\boxed{Q_{ij} = \int d^3x'\,\rho(\vec{x}')\left[3x_i' x_j' - r'^2\delta_{ij}\right]}$$

By analogy, $Y_{21} = -\sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta\, e^{i\varphi} =$

$$= -\sqrt{\frac{15}{8\pi}}\,\frac{z}{r^2}(x+iy) \Rightarrow q_{21} = -\sqrt{\frac{15}{8\pi}}\int d^3x'\,\rho(\vec{x}')\cdot$$

$$\cdot z'(x'-iy') = -\frac{1}{3}\sqrt{\frac{15}{8\pi}}\left(Q_{13} - i\,Q_{23}\right)$$

One can also show that $q_{20} = \frac{1}{2}\sqrt{\frac{5}{4\pi}}\,Q_{33}$.

As $Y_{\ell,-m}(\theta,\varphi) = (-1)^m Y^*_{\ell m}(\theta,\varphi) \Rightarrow \boxed{q_{\ell,-m} = (-1)^m q^*_{\ell m}}$

can use to obtain other $q_{\ell,-m}$'s.

Using the found multipole moments we get

$$\Phi(\vec{x}) = \frac{1}{\varepsilon_0}\frac{q_{00}}{r}Y_{00} + \frac{1}{3\varepsilon_0}\frac{1}{r^2}\left(q_{11}Y_{11} + q_{10}Y_{10} + q_{1,-1}Y_{1,-1}\right) +$$

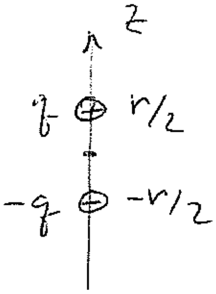
$$+ \ldots = \frac{1}{4\pi\varepsilon_0}\frac{q}{r} + \frac{1}{\cancel{3}\varepsilon_0}\frac{\cancel{3}}{8\pi}\frac{1}{r^2}\left(+(p_x - ip_y)\cdot\frac{x+iy}{r} + \right.$$

$$\left. + (p_x + ip_y)\frac{x-iy}{r} + 2p_z\frac{z}{r}\right) + \ldots \Rightarrow$$

$$\Phi(\vec{x}) = \frac{1}{4\pi\varepsilon_0}\left[\frac{q}{r} + \frac{\vec{p}\cdot\vec{x}}{r^3} + \frac{1}{2}\sum_{i,j} Q_{ij}\,\frac{x_i x_j}{r^5} + \dots\right]$$

can also be derived.

Examples:



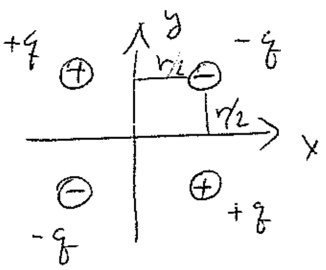
dipole: $q = 0$.

$$\vec{p} = q\,\frac{\vec{r}}{2} - (-q)\left(-\frac{\vec{r}}{2}\right) = q\vec{r} \ \checkmark \text{ok.}$$

$$Q_{zz} = 3\left(\frac{r}{2}\right)^2 q - 3\left(\frac{r}{2}\right)^2 q - \left(\frac{r}{2}\right)^2 q + \left(\frac{r}{2}\right)^2 q = 0$$

all $Q_{ij} = 0, \ \dots$

quadrupole:



$=$ 

$q = 0$

$\Rightarrow \vec{p} = 0$

$$Q_{xx} = -q\cdot\left(3\left(\frac{r}{2}\right)^2 - \frac{r^2}{2}\right)2 + q\left(+3\left(\frac{r}{2}\right)^2 - \frac{r^2}{2}\right) = 0$$

$$Q_{xy} = -q\cdot 3\left(\frac{r}{2}\right)^2\cdot 2 + q\cdot 3\left(\frac{r}{2}\right)\left(-\frac{r}{2}\right)\cdot 2 = -3qr^2 \neq 0.$$

non-zero quadrupole moment.

$Q_{yy} = 0.$

If we know Φ, we know $\vec{E} = -\vec{\nabla}\Phi \Rightarrow$

$$\Rightarrow \vec{E} = \frac{1}{4\pi\varepsilon_0}\left[\frac{q}{r^3}\vec{r} + \underbrace{\frac{3\hat{n}(\vec{p}\cdot\hat{n}) - \vec{p}}{r^3}}_{\text{dipole's electric field}} + \dots\right]$$

$$\hat{n} = \frac{\vec{r}}{r}.$$